UNIT - I

PARTIAL DIFFERENTIAL EQUATIONS

Any equations containing partial derivatives is called partial Differential equation.

Example. $\boxed{\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial z}{\partial x} + x + y = 0}$

The order of the partial Differential equation (PDE) is the highest derivative in that equation

Degree

The Degree of a partial Differential equation is the power of highest derivative in that equation.

Notations:

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} \text{ (or) } \frac{\partial^2 z}{\partial y \partial x}$$

(Formation of partial differential equations) Elimination of arbitrary function, from the relation containing the variables z, x, y, a, b.

Obtain the partial Differential equations by eleminating constands a $ b from $(x-a)^2+(y-b)^2+z^2=1$

Soln $(x-a)^2+(y-b)^2+z^2=1$ —— ①

Diff. w.r. to 'x'

$2(x-a)+2zp=0$

$\cancel{2}(x-a)=-\cancel{2}zp$

$x-a=-zp$ —— ②

Diff. w.r. to 'y'

$2(y-b)+2zq=0$

$2(y-b)=-2zq$

$y-b=-zq$ —— ③

Sub the ② $ ③ in eqn ①

$z^2p^2+z^2q^2+z^2=1$

$\boxed{z^2(p^2+q^2+1)=1}$

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ELIMINATION OF ONE ARBITRARY FUNCTION

Method. $\phi(u,v) = 0$

From the PDE by elemenating the arbitrary function from $\phi\left[z^2 - xy, \; x/z\right] = 0$

This is in one form of $\phi(u,v) = 0$

$u = z^2 - xy \qquad v = x/z \qquad \frac{vDu - uDv}{v^2}$

$$\frac{\partial u}{\partial x} = 2zp - y \qquad \frac{\partial v}{\partial x} = \frac{z - xp}{z^2}$$

$$\frac{\partial u}{\partial y} = 2zq - x \qquad \frac{\partial v}{\partial y} = x\left(-q/z^2\right) = -\frac{xq}{z^2}$$

Elimination of ϕ gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2zp - y & \frac{z - xp}{z^2} \\ 2zq - x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$(2zp - y)\left(-\frac{xq}{z^2}\right) - (2zq - x)\left(\frac{z - xp}{z^2}\right) = 0$$

$$(2zp - y)(-xq) - (2zq - x)(z - xp) = 0$$

$$-2xzpq + xyq - \left[2z^2q + 2zxpq - xz + x^2p\right] = 0$$

$$-\cancel{2zxpq} + xyq - 2z^2q + \cancel{2zxpq} + xz - x^2p = 0$$

$$xyq - 2z^2q + xz - x^2p = 0$$

$$(xy - 2z^2)q - x^2p = -xz$$

Method 2 : The given relation not in the form of

$\phi(u,v)=0$

1. Form the PDE by eleminating the orbitrary function

$z = f(x^2+y^2) + x+y$

Soln

$$z = f(x^2+y^2) + x+y \quad \text{--- (1)}$$

Diff. w.r. to 'x'

$$p = f'(x^2+y^2) \cdot 2x + 1$$

$$p-1 = f'(x^2+y^2) \cdot 2x \quad \text{--- (2)}$$

Diff. w.r. to 'y'

$$q = f'(x^2+y^2) \cdot 2y + 1$$

$$q-1 = f'(x^2+y^2) \cdot 2y \quad \text{--- (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p-1}{q-1} = \frac{x}{y}$$

$$(p-1)\, y = (q-1)\, x$$

$$py - y = qx - x$$

$$\boxed{py - qx = y - x}$$

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From the PDE by eleminating the arbitrary function f & g in $z = f(x^3+2y) + g(x^3-2y)$

Soln

$$z = f(x^3+2y) + g(x^3-2y)$$

$$p = f'(x^3+2y) \cdot 3x^2 + g'(x^3-2y) \cdot 3x^2$$

$$q = f'(x^3+2y)\cdot 2 + g'(x^3-2y)(-2)$$

$$r = f'(x^3+2y)\, 6x + 3x^2\, f''(x^3+2y)\cdot 3x^2 + g'(x^3-2y)\, 6x + 3x^2\, g''(x^3-2y)\, 3x^2$$

$$r = 6x\, f'(x^3+2y) + 9x^4 f''(x^3+2y) + 6x\, g'(x^3-2y) + 9x^4\, g''(x^3-2y)$$

$$r = 6x\left[f'(x^3+2y) + g'(x^3-2y)\right] + 9x^4\left[f''(x^3+2y) + g''(x^3-2y)\right]$$

$$t = 4f''(x^3+2y) + 4g''(x^3-2y)$$

$$r = \overset{2}{\cancel{6}}x\; p/\cancel{3}x^2 + 9x^4\, t/4$$

$$\boxed{r = 2p/x + 9x^4\, t/4}$$

FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

Type 1:

$F(p, q) = 0$

1. Solve $p+q = pq$

$p+q = pq$ ——— ①

Let $z = ax+by+c$ ——— ② be a soln of ①.

Diff ② w.r.to 'x' & 'y'

$p = a$

$q = b$

subs these in ①

$a+b = ab$

$b - ab = -a$

$b(1-a) = -a$

$b = \frac{-a}{1-a}$

$b = a/a-1$

subs these value in ②

$z = ax + \frac{a}{a-1} y + c$ ——— ③

which is C.I

TO Find S.I (Single Integral)

Diff ③ par with respect to 'c'

$0 = 1$

Which is impossible

Hence there is no S.I

To Find G.I (General Integral)

Put $c = \phi(a)$ in ③

$$z = ax + \left(\frac{a}{a-1}\right) y + \phi(a) \quad \text{——} \; ④$$

Diff ④ partialy w.r.to 'a'

$$0 = x + y \left[\frac{(a-1) - a}{(a-1)^2}\right] + \phi'(a) \quad \text{——} \; ⑤$$

Eliminating 'a' from ④ & ⑤ we get G.I

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Type 2:

$z = px + qy + f(p, q)$ is Clairaut's form

1. Solve $z = px + qy + \sqrt{1+p^2+q^2}$

Put $p = a$

$q = b$ in (1)

$z = ax + by + \sqrt{1+a^2+b^2}$ ——— (2)

Which is C.I

To find S.I

Diff (2) partialy w.r.to 'a'

$0 = x + \frac{1}{2}(1+a^2+b^2)^{-1/2}\ \cancel{2}a$

$x = \dfrac{-a}{\sqrt{1+a^2+b^2}}$ ——— (3)

Diff (2) partialy w.r.to 'b'

$0 = y + \frac{1}{2}(1+a^2+b^2)^{-1/2} \cdot 2b$

$0 = y + (1+a^2+b^2)^{-1/2}\ b$

$\boxed{y = \dfrac{-b}{\sqrt{1+a^2+b^2}}}$ ——— (4)

$$③^2 + ④^2 \Rightarrow x^2+y^2 = \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}$$

$$= \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1-\left(\frac{a^2+b^2}{1+a^2+b^2}\right)$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$1+a^2+b^2 = \frac{1}{1-x^2-y^2}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1+x^2-y^2}} \quad \text{——} ⑤$$

From eq no ③

$$a = -x\sqrt{1+a^2+b^2}$$

$$\boxed{a = \frac{-x}{\sqrt{1-x^2-y^2}}} \quad \text{——} ⑥$$

From ④ $\quad b = -y\sqrt{1+a^2+b^2}$

$$\boxed{b = \frac{-y}{\sqrt{1-x^2-y^2}}} \quad \text{——} ⑦$$

sub ⑤, ⑥ & ⑦ in ②

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$\boxed{x^2+y^2+z^2=1}$$

$$z = px + qy + \sqrt{16+p^2+q^2} \quad \text{--- (1)}$$

$$p = a$$

$$q = b$$

$$z = ax + by + \sqrt{16+a^2+b^2} \quad \text{--- (2)}$$

To find S.I

Diff (2) partialy w.r.to 'a'

$$x = \frac{-a}{\sqrt{16+a^2+b^2}} \quad \text{--- (3)}$$

Diff (2) part w.r.to 'b'

$$y = \frac{-b}{\sqrt{16+a^2+b^2}} \quad \text{--- (4)}$$

(3)^2 + (4)^2

$$x^2+y^2 = \left(\frac{a}{\sqrt{16+a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{16+a^2+b^2}}\right)^2$$

$$x^2+y^2 = \frac{a^2+b^2}{16+a^2+b^2}$$

$$1-x^2+y^2 = 1 - \left(\frac{a^2+b^2}{16+a^2+b^2}\right)$$

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$$1-x^2-y^2 = \frac{16+a^2+b^2-a^2-b^2}{16+a^2+b^2}$$

$$16+a^2+b^2 = \frac{16}{1-x^2-y^2}$$

$$\sqrt{16+a^2+b^2} = \frac{4}{\sqrt{1-x^2-y^2}} \quad \text{—— (5)}$$

From eqn no (1) take the value of

$$a = -x\sqrt{16+a^2+b^2}$$

$$a = \frac{-4x}{\sqrt{1-x^2-y^2}} \quad \text{—— (6)}$$

$$b = \frac{-4y}{\sqrt{1-x^2-y^2}} \quad \text{—— (7)}$$

(5), (6), (7) is (2)

$$z = \frac{-4x^2}{\sqrt{1-x^2-y^2}} - \frac{4y^2}{\sqrt{1-x^2-y^2}} + \frac{4}{\sqrt{1-x^2-y^2}}$$

$$= \frac{4-4x^2-4y^2}{\sqrt{1-x^2-y^2}}$$

$$= \frac{4(1-x^2-y^2)}{\sqrt{1-x^2-y^2}}$$

$$z = 4\sqrt{1-x^2-y^2}$$

$$z^2 = 16(1-x^2-y^2)$$

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Type (iii)

case (i) $f(p, q, z) = 0$

Solve $z = 1 + p^2 + q^2$

Soln $z = 1 + p^2 + q^2$ —— (1)

$z = f(x + ay)$ be a soln of (1)

put $x + ay = u$

$z = f(u)$

Diff (1) partialy w.r.to 'x' & 'y'

$$p = \frac{dz}{du}$$

$$q = a\frac{dz}{du}$$

Sub the value in (1)

$$z = 1 + \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$= 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z - 1 = \left(\frac{dz}{du}\right)^2 (1 + a^2)$$

$$\sqrt{1+a^2}\ \frac{dz}{dx} = \sqrt{z-1}$$

$$\sqrt{1+a^2} \int (z-1)^{-1/2}\, dz = \int du$$

$$\sqrt{1+a^2}\ \frac{(z-1)^{1/2}}{1/2} = u+c$$

$$\int \{f(x)\}^n f'(x)dx$$

$$2\cdot\sqrt{1+a^2}\ \sqrt{z-1} = x+ay+c$$

$$= \frac{\{f(x)\}^{n+1}}{n+1}$$

Which is C.I

TYPE IV

Separable Equations $f(x,p) = g(y,q)$

Solve $p^2 y (1+x^2) = q x^2$

Given $p^2 y (1+x^2) = q x^2$ —— (1)

$$p^2 \left(\frac{1+x^2}{x^2} \right) = \frac{q}{y} = a.$$

$$p^2 \left(\frac{1+x^2}{x^2} \right) = a$$

$$p^2 = \frac{a x^2}{1+x^2}$$

$$p = \frac{\sqrt{a}\, x}{\sqrt{1+x^2}}$$

$$\frac{q}{y} = a$$

$$\boxed{q = ay}$$

WKT $dz = p\,dx + q\,dy$

$$dz = \frac{\sqrt{a}\, x}{\sqrt{1+x^2}} dx + ay\,dy$$

$$\int dz = \frac{\sqrt{a}}{2} \int (1+x^2)^{-1/2}\, 2x\,dx + a \int y\,dy$$

$$z = \frac{\sqrt{a}}{2} \frac{(1+x^2)^{1/2}}{1/2} + \frac{ay^2}{2} + c$$

$$z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + c.$$

Type n:

$$f(px^m, qy^n) = 0$$

(or)

$$f(px^m, qy^n, z) = 0$$

case(i)

If $m \neq 1$

Then $X = x^{1-m}$

If $n \neq 1$

Then $Y = y^{1-n}$

case(ii) If $m = 1$

Then $X = \log x$

If $n = 1$

Then $Y = \log y$

$$p^2 + x^2y^2q^2 = x^2z^2$$

Soln:

$$p^2 + x^2y^2q^2 = x^2z^2 \quad \text{--- (1)}$$

$\div$ by x^2 $\quad \dfrac{p^2}{x^2} + q^2y^2 = z^2$

$$(px^{-1})^2 + (qy)^2 = z^2 \quad \text{--- (2)}$$

$m = -1 \qquad n = 1$

$X = x^{1-n}$ $\quad Y = \log y$

$X = x^{n}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$$

$p = P x^{-n}$

$\boxed{p x^{n} = P}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$q = Q/y$

$\boxed{q y = Q}$

Sub the value of px' & qy' in (A)

$A P^2 + Q^2 = z^2$ ——— (1)

Let $z = f(X + aY)$ be soln eqn (1)

Put $X + aY = u$

$z = f(u)$

Diff partial w.r.to X' & Y'

$P = \frac{dz}{du}$; $Q = \frac{dz}{du} a$

Sub the value in (1)

$$A\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{dx}\right)^2 (1+a^2) = z^2$$

$$\frac{dz}{dx}\sqrt{1+a^2} = z$$

$$\sqrt{1+a^2}\,\frac{dz}{z} = dx$$

$$\sqrt{1+a^2}\int \frac{1}{z}\,dz = \int dx$$

$$\sqrt{1+a^2}\,\log z = u + c$$

$$\sqrt{1+a^2}\,\log z = X + aY + c$$

$$\sqrt{1+a^2}\,\log z = x^2 + a\log y + c$$

$$\boxed{\sqrt{1+a^2}\,\log z = x^2 + a\log y + c}$$ (Ans)

Type $\overline{\text{VI}}$

$$F(pz^k, qz^k) = 0$$

case (i) If $k = -1$

$$Z = \log z$$

case (ii) If $k \neq -1$

$$Z = z^{k+1}$$

Solve $(x+pz)^2 + (y+qz)^2 = 1$

$$(x+pz)^2 + (y+qz)^2 = 1 \quad \text{———} \quad (*)$$

$$k = 1$$

$$Z = z^{k+1}$$
$$= z^{1+1}$$

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$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z} \cdot \frac{\partial Z}{\partial x}$$

$$P = 2Zp \qquad \boxed{pZ = P/2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z} \cdot \frac{\partial Z}{\partial y}$$

$$Q = 2Zq$$

$$\boxed{Zq = \frac{Q}{2}}$$

Sub the values of px^{-1} & qy in (*)

$$(x + P/2)^2 + (y + Q/2)^2 = 1$$

$$(x + P/2)^2 = 1 - (y + Q/2)^2 = a^2$$

$$(x + P/2)^2 = a^2$$

$$x + P/2 = a$$

$$P/2 = a - x$$

$$\boxed{P = 2(a - x)}$$

$$1 - (y + Q/2)^2 = a^2$$

$$(y + Q/2)^2 = 1 - a^2$$

$$y + Q/2 = \sqrt{1 - Q^2}$$

$$Q/2 = \sqrt{1 - Q^2} - y$$

$Q = 2\left[\sqrt{1-a^2} - y\right]$

WKT $dz = p\,dx + Q\,dy$

$\int dz = 2\int (a-x)\,dx + 2\int \left(\sqrt{1-a^2} - y\right) dy + c$

$z = 2\left(ax - x^2/2\right) + 2\sqrt{1-a^2}\,y - 2y^2/2 + c$

$z = 2ax - x^2 + 2\sqrt{1-a^2}\,y - y^2 + c$

Solve $z^2(p^2+q^2) = x^2+y^2$

Soln $p^2z^2 + q^2z^2 = x^2+y^2$ —— (*)

$(pz)^2 + (qz)^2 = x^2+y^2$

$k = 1$

$Z = z^{k+1}$

$Z = z^2$

$$\frac{dZ}{dx} = \frac{dZ}{dz}\cdot\frac{dz}{dx}$$

$P = 2zp$

$pz = P/2$

$$\frac{dZ}{dy} = \frac{dZ}{dz}\cdot\frac{dz}{dy}$$

$Q = \cancel{2}zq$

$zq = Q/2$

$\cancel{Qz} \quad Qz$

$pz,\ Qz$ in eq (*)

$$P^2 \left(y + Q^2/4\right) = x^2 + 4^2$$

$$P^2/4 - x^2 = y^2 - Q^2/4$$

$$P^2/4 - x^2 = a$$

$$P^2/4 = a - x^2$$

$$P^2 = 4(a - x^2)$$

$$P = 2\sqrt{a - x^2}$$

$$y - Q^2/4 = a$$

$$Q^2/4 = y^2 - a$$

$$Q^2 = 4(y^2 - a)$$

$$Q = 2\sqrt{y^2 - a}$$

W.K.T $dz = P\,dx + Q\,dy$

$$\int dz = 2\int \sqrt{a - x^2}\,dx + 2\int \sqrt{y^2 - a^2}\,dy$$

$$= \frac{2(a - x^2)^{3/2}}{3/2} + 2\,\frac{(y^2 - a)^{3/2}}{3/2} + c$$

$$z = 4/3\,(a - x^2)^{3/2} + 4/3\,(y^2 - a)^{3/2} + c.$$

LAGRANGE'S LINEAR EQUATIONS

Any equations is of the form $Pp+Qq=R$, where P, Q & R are function of x, y, z is called Lagrande's linear equation

Grouping method

1. Solve $(1-x)p+(2-y)q = 3-z$

$(1-x)p+(2-y)q = 3-z$

which is Lagrande's equations

$P = (1-x) \quad Q = (2-y) \quad R = 3-z$

The A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{1-x} = \frac{dy}{2-y} = \frac{dz}{3-z}$ ——— (1)

consider the 1^{st} & 2^{nd} ratio in (1)

$$\frac{dx}{1-x} = \frac{dy}{2-y}$$

$$\frac{dx}{x-1} = \frac{dy}{y-2}$$

$$\int \frac{dx}{x-1} = \int \frac{dy}{y-2}$$

$$\log(x-1) = \log(y-2) + \log a$$

$$\log(x-1) - \log(y-2) = \log a.$$

$$\log \frac{x-1}{y-2} = \log a$$

$$\frac{x-1}{y-2} = a$$

$$u = \frac{x-1}{y-2}$$

consider the last 2 ratio's in ①

$$\frac{dy}{2-y} = \frac{dz}{3-z}$$

$$\frac{dy}{y-2} = \frac{dz}{z-3}$$

$$\int \frac{dy}{y-2} = \int \frac{dz}{z-3}$$

$$\log(y-2) = \log(z-3) + \log b$$

$$\log\left(\frac{y-2}{z-3}\right) = \log b$$

$$\frac{y-2}{z-3} = b$$

$$v = \frac{y-2}{z-3}$$

The G.S is

$$\phi\left(\frac{x-1}{y-2}, \frac{y-2}{z-3}\right) = 0 \quad \text{(Ans)}$$

METHOD OF MULTIPLIERS

6 Solve $x(y-z)p + y(z-x)q = z(x-y)$

$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y)$

The A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

choose multiplies $1, 1, 1$

Each ratio in ① $= \frac{dx + dy + dz}{0}$

$dx + dy + dz = 0$

$\int$ing $x + y + z = a$

$u = x + y + z$

choose another multiplies $1/x, 1/y, 1/z$

Each ratio in ① $= \frac{1/x\, dx + 1/y\, dy + 1/z\, dz}{0}$

$1/x\, dx + 1/y\, dy + 1/z\, dz = 0$

$\int$ing $\log x + \log y + \log z = \log b$

$\log(xyz) = \log b$

$b = xyz$

$v = xyz$

$\phi(x+y+z, xyz) = 0$

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7 Solve $(mz - ny)p + (nx - lz)q = ly - mx$

$P = mz - ny \quad Q = nx - lz \quad R = ly - mx$

The A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

choose Multiplies x, y, z

Each ratio in ① $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a/2$$

Sing $x^2 + y^2 + z^2 = a$

$u = x^2 + y^2 + z^2$

Choosing another Multiplies l, m, n

Each ratio in ① $= \dfrac{l\,dx + m\,dy + n\,dz}{0}$

$l\,dx + m\,dy + n\,dz = 0$

Sing $lx + my + nz = 0$

$v = lx + my + nz$

The G.S is

$\phi(x + y + z,\ lx + my + nz) = 0$

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HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS

General form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

(or)

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = F(x, y)$$

To find complementary function

case(i) If $m_1, m_2, \ldots\ldots m_n$ are real & distinct

$$C.F = \phi_1 (y + m_1 x) + \phi_2 (y + m_2 x) + \phi_3 (y + m_3 x) + \cdots + \phi_n (y + m_n x)$$

case(ii)

$m_1 = m_2 \qquad m_3, m_4 \ldots$ are different

$$C.F = \phi_1 (y + mx) + x \phi_2 (y + mx) + \phi_3 (y + m_3 x) + \cdots + \phi_n (y + m_n x)$$

case(iii)

$m_1 = m_2 = m_3 \qquad m_4, m_5 \ldots$ are diffrent

$$C.F = \phi_1 (y + mx) + x \phi_2 (y + mx) + x^2 \phi_3 (y + mx) + \phi_4 (y + m_4 x) + \cdots + \phi_n (y + m_n x)$$

1 Solve $4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$

Soln

$$4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$$

$$(4D^2 - 12DD' + 9D'^2)z = 0$$

The A.E is $4m^2 - 12m + 9 = 0$

$$4m^2 - 6m - 6m + 9 = 0$$

$$2m(2m-3) - 3(2m-3) = 0$$

$$(2m-3)(2m-3) = 0$$

$$m = 3/2, 3/2$$

$$C.F = \phi_1(y + 3/2\,x) + x\,\phi_2(y + 3/2\,x)$$ (Ans)

Type 2:

R.H.S = $\sin ax \sin by$

(or)

$\cos ax \cos by$

$D^2 \to -a^2$, $D'^2 \to -b^2$

1) $(D^2 - D'^2) z = \sin 2x \sin 3y$

The A.E is

$m^2 - 1 = 0 \quad m^2 = 1$
$m = 1, -1$

$C.F = \phi_1 (y+x) + \phi_2 (y-x)$

$P.I = \dfrac{1}{D^2 - D'^2} \sin 2x \sin 3y$

$a = 2$
$b = 3$
$D^2 \to -4$
$D'^2 \to -9$

$= \dfrac{1}{5} \sin 2x \sin 3y$

The G.S is $z = C.F + P.I$

$z = \phi_1 (y+x) + \phi_2 (y-x) + 1/5 \sin 2x \sin 3y.$

Solve $(D^2 - 4D'^2) z = \cos 3x \cos 5y$

The A.E is

$(m^2 - 4) = 0$

$m^2 = 4$

$m = 2, 2$

$C.F = \phi_1 (y+2x) + x\phi_2 (y+2x)$

$P.I = \dfrac{1}{D^2 - 4D'^2} \cos 3x \cos 5y$

$= \dfrac{1}{91} \cos 3x \cos 5y$

The G.S is

$z = C.F + P.I$

$z = \phi_1 (y+2x) + x\phi_2 (y+2x) + 1/91 \cos 3x \cos 5y$

Type 3:

R.H.S = $\sin(ax+by)$ or $\cos(ax+by)$

$D^2 \to -a^2$, $D'^2 \to -b^2$, $DD' \to -ab$

1 Solve $(D^2 - 3DD' + 2D'^2)z = \cos(x+2y)$

Soln

The A.E is

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2, 1$

$C.F = \phi_1(y+x) + \phi_2(y+2x)$

$$P.I = \frac{1}{D^2 - 3DD' + 2D'^2} \cos(x+2y)$$

$a = 1$
$b = 2$
$D^2 \to -1$
$D'^2 \to -4$
$DD' \to -2$

$$= \frac{1}{(-1) - 3(-2) + 2 \times (-4)} \cos(x+2y)$$

$P.I = -\frac{1}{3} \cos(x+2y)$

The G.S is $z = C.F + P.I$

$z = \phi_1(y+x) + \phi_2(y+2x) - \frac{1}{3}\cos(x+2y)$

2 Solve $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 8(\sin(x+3y))$

Soln The A.E is

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1, 2$

$C.F = \phi_1(y+x) + \phi_2(y+2x)$

$a = 1$
$b = 3$

$$P.I = \frac{1}{D^2 - 3DD' + 2D'^2} \, 8\sin(x+3y)$$

$$= \frac{1}{(-1) - 3(-3) + 2(-9)} \, 8\sin(x+3y)$$

$$= -\frac{1}{\cancel{10}} \, \cancel{8} \sin(x+3y)$$

$$P.I = -4/5 \sin(x+3y)$$

The G.S $z = C.F + P.I$

$$z = \phi_1(y+x) + \phi_2(y+2x) - 4/5 \sin(x+3y)$$

TYPE 4:

$R.H.S = x^n y^m$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

Solve $(D^2 + 4DD' - 5D'^2) z = x + y^2 + \pi$

The A.E is

$m^2 + 4m - 5 = 0$

Type 5 :

$$R.H.S = e^{ax+by} \phi(x, y)$$

$$D \to D+a \qquad D' \to D'+b$$

Solve the equation $(D^2 - 2DD' + D'^2)z = x^2y^2 e^{x+y}$

Soln The A.E is

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$C.F = \phi_1(y+x) + x\phi_2(y+x)$$

$a = 1$
$b = 1$
$D \to D+1$
$D' \to D'+1$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} x^2y^2 e^{x+y}$$

$$= e^{x+y} \frac{1}{(D+1)^2 - 2(D+1)(D'+1) + (D'+1)^2} x^2y^2$$

$$= e^{x+y} \frac{1}{D^2 + 2D + 1 - 2(DD' + D + D' + 1) + (D'^2 + 2D' + 1)} x^2y^2$$

$$= e^{x+y} \frac{1}{D^2 + \cancel{2D} + \cancel{1} - 2DD' - \cancel{2D} - \cancel{2D'} - \cancel{2} + D'^2 + \cancel{2D'} + \cancel{1}} x^2y^2$$

$$= e^{x+y} \frac{1}{D^2 + D'^2 - 2DD'} x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\frac{1}{1+\left(\frac{D'^2}{D^2}-\frac{2D'}{D}\right)}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[1+\left(\frac{D'^2}{D^2}-\frac{2D'}{D}\right)\right]^{-1}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[1-\left(\frac{D'^2}{D^2}-\frac{2D'}{D}\right)+\frac{4D'^2}{D^2}\right]x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[x^2y^2-\frac{2x^2}{D^2}+\frac{2}{D}(2x^2y)+\frac{4}{D^2}(2x^2)\right]$$

$$= e^{x+y}\frac{1}{D^2}\left[x^2y^2-\cancel{2}\frac{x^4}{\cancel{12}_6}+4\frac{x^3}{3}y+\cancel{8}^{2}\frac{x^4}{\cancel{12}_3}\right]$$

$$= e^{x+y}\frac{1}{D^2}\left[x^2y^2-\frac{x^4}{6}+\frac{4}{3}x^3y+2/3\,x^4\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}-\frac{x^6}{180}+4/3\,\frac{x^5y}{\cancel{20}_5}+2/3\,\frac{x^6}{\cancel{30}_{15}}\right]$$

$$P.I = e^{x+y}\left[\frac{x^4y^2}{12}-\frac{x^6}{180}+\frac{x^5y}{15}+\frac{x^6}{45}\right]$$

The G.S is

$$z = C.F + P.I$$

$$z = \phi_1(y+x)+x\phi_2(y+x)+e^{x+y}\left[\frac{x^4y^2}{12}-\frac{x^6}{180}+\frac{x^5y}{15}+\frac{x^6}{45}\right] \text{(Ans)}$$

Type 6

R.H.S : $y \cos x$ (or) $y \sin x$

$\vdots$

Solve $(D^2 + DD' - 6D'^2)z = y \cos x$

Soln

The A.E is

$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$C.F = \phi_1(y+2x) + \phi_2(y-3x)$

$$P.I = \frac{1}{D^2+DD'-6D'^2}\, y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos x$$

$y = a-2x$

$a \to y+2x$

$$= \frac{1}{D+3D'}\left[\int (a-2x)\cos x\, dx\right]$$

$$= \frac{1}{D+3D'}\left[(a-2x)\sin x - (-2)(-\cos x)\right]_{a \to y+2x}$$

$$= \frac{1}{D+3D'}(y\sin x - 2\cos x)$$

$$= \left[\int (a+3x)\sin x\, dx\right]_{a\to y-3x} - 2\int \cos x\, dx$$

$y \to a+3x$

$a \to y-3x$

$$= \left[(a+3x)(-\cos x) - 3(-\sin x)\right]_{a \to y-3x} - 2\sin x$$

$$= -y\cos x + 3\sin x - 2\sin x$$

$$P.I = \sin x - y\cos x$$

The G.S is

$$z = C.F + P.I$$

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$$

MIXED TYPE:

1 Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x-y)$

Sol $(D^2 - DD' - 20D'^2)z = e^{5x+y} \sin(4x-y)$

The A.E is

$$m^2 - m - 20 = 0$$

$$(m-5)(m+4) = 0$$

$$m = 5, -4$$

$$C.F = \phi_1(y+5x) + \phi_2(y-4x)$$

$$P.I = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$

a =
b ·
D ·
D'

$$= \frac{1}{25 - (5)(1) - 20(1)} e^{5x+y}$$

$$= \frac{1}{25 - 5 - 20} e^{5x+y}$$

$$= \frac{1}{25 - 25} e^{5x+y}$$

$$= x \frac{1}{2D - D'} e^{5x+y}$$

$$= x \frac{1}{10 - 1} e^{5x+y}$$

$$P.I_1 = x/9 \, e^{5x+y}$$

$a = 4$
$b = -1$
$D^2 \to -16$
$D'^2 \to -1$
$DD' \to 4$

$$P.I = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$$= \frac{1}{-16-(4)-20(-1)} \sin(4x-y)$$

$$= \frac{1}{-16-4+20} \sin(4x-y)$$

$$= x \frac{1}{2D-D'} \sin(4x-y)$$

$$= x \frac{2D+D'}{(2D-D')(2D+D')} \sin(4x-y)$$

$$= x \frac{2D+D'}{4D^2-D'^2} \sin(4x-y)$$

$$= x \frac{2D+D'}{4\times-16-(-1)} \sin(4x-y)$$

$$= x \frac{2D+D'}{-64+1} \sin(4x-y)$$

$$= -x/63 \, (2D+D') \sin(4x-y)$$

$$= -x/63 \left[2D(\sin(4x-y)) + D'(\sin(4x-y))\right]$$

$$= -x/63 \left[8\cos(4x-y) - \cos(4x-y)\right]$$

$$= -x/\cancel{63}_{9} \, \cancel{7} \cos(4x-y)$$

$$P.I_2 = -x/9 \cos(4x-y)$$

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$\therefore$ The G.S is $z = C.F + P.I$

$$z = \phi_1(y+5x) + \phi_2(y-4x) + x/9\, e^{5x+y} - x/9 \cos(4x$$

2 Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x+y)$

The A.E is

$m^3 + m^2 - m - 1 = 0$

$m = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -1 & -1 \\ & 0 & -1 & 0 & 1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$m = -1, -1, 1$

$C.F = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x)$

$$P.I = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^{2x+y}$$

$a = 2$

$b = 1$

$D \to 2$

$D' \to 1$

$$= \frac{1}{8+4-2-1} e^{2x+y}$$

$$= \frac{1}{12-3} e^{2x+y}$$

$P.I_1 = 1/9\, e^{2x+y}$

$$P.I_2 = \frac{1}{D^3+D^2D' - DD'^2 - D'^3} \cos(x+y)$$

$a=1$
$b=1$
$D^2 \rightarrow -1$
$D'^2 \rightarrow -1$
$DD' \rightarrow -1$

$$= \frac{1}{-D-D'+D+D'} \cos(x+y)$$

$$= x \frac{1}{3D^2+2DD'-D'^2} \cos(x+y)$$

$$= x \frac{1}{3(-1)+2(-1)-(-1)} \cos(x+y)$$

$$= x \frac{1}{-3-2+1} \cos(x+y)$$

$$P.I_2 = -x/4 \cos(x+y)$$

The G.S is $z = C.P + P.I_1 + P.I_2$

$$z = \phi_1(y-x) + x\,\phi_2(y-x) + \phi_3(y+x) + 1/9\, e^{2x+y} - x/4 \cos(x+y)$$

NON HOMOGENEOUS PDE

To find Complementary function:

Case(i)

$$(D - mD' - \alpha)z = 0$$

$$C.F = e^{\alpha x}\phi_1(y+mx)$$

Case(ii)

$$(D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)z = 0$$

$$C.F = e^{\alpha_1 x}\phi_1(y+m_1x) + e^{\alpha_2 x}\phi_2(y+m_2x)$$

Case(iii)

$$(D - mD' - \alpha)^r z = 0$$

$$C.F = e^{\alpha x}\phi_1(y+mx) + x e^{\alpha x}\phi_2(y+mx) + \dots + x^{r-1}e^{\alpha x}\phi_r(y+mx)$$

Solve $(D^2 - DD' + D' - 1)z = 0$

$$(D^2 - DD' + D' - 1)z = 0$$

$$(D-1)(D-D'+1)z = 0$$

$D^2 - DD' + D' - 1$
$= D^2 - 1 - DD' + D'$
$= (D-1)(D+1) - D'(D-1)$
$= (D-1)[D-D'+1]$

$m_1 = 0 \qquad m_2 = 1$

$\alpha_1 = 1 \qquad \alpha_2 = -1$

$$C.F = e^{x}\phi_1(y) + e^{-x}\phi_2(y+x)$$

Solve $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$

Soln

$$(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$$

$$[(D-D')^2 - 3D + 3D' + \underline{3-1}]z = (e^{3x})^2 + 4e^{3x}e^{-2y} + 4(e^{-2y})$$

$$[(D-D')^2 - 3(D-D'-1) - 1]z = e^{6x} + 4e^{3x-2y} + 4e^{-4y}$$

$$[\underline{(D-D')^2 - 1} - 3(D-D'-1)]z = e^{6x} + 4e^{3x-2y} + 4e^{-4y}$$

$$[(\underline{D-D'}-1)(D-D'+1) - 3(\underline{D-D'}-1)]z = e^{6x} + 4e^{3x-2y} + 4e^{-4y}$$

$$[(D-D'-1)(D-D'-2)]z = e^{6x} + 4e^{3x-2y} + 4e^{-4y}$$

$m_1 = 1 \qquad m_2 = 1$

$\alpha_1 = 1 \qquad \alpha_2 = 2$

$$C.F = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x)$$

$$P.I_1 = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} e^{6x}$$

$$= \frac{1}{36 - 18 + 2} e^{6x}$$

$$P.I_1 = \frac{1}{20} e^{6x}$$

$$P.I_2 = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} 4e^{3x-2y}$$

$$= 4\frac{1}{\cancel{9} + 12 + 4 - \cancel{9} - 6 + 2} e^{3x-2y}$$

$$= 4\frac{1}{27 - 15} e^{3x-2y}$$

$$= 4\,\frac{1}{12}\, e^{3x-2y}$$

$$P.I_2 = \frac{1}{3} e^{3x-2y}$$

$$P.I_3 = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} 4e^{-4y}$$

$$= 4 \frac{1}{16 - 12 + 2} e^{-4y}$$

$$= \cancel{4}^{2} \frac{1}{\cancel{6}_{3}} e^{-4y}$$

$$P.I_3 = 2/3 \; e^{-4y}$$

$$z = C.F + P.I_1 + P.I_2 + P.I_3$$

$$z = e^{x}\phi_1 (y+x) + e^{2x}\phi_2 (y+x) + 1/20 \, e^{6x} + 1/3 \, e^{3x-2y} + 2/3 \, e^{-4y}$$

UNIT – II

Dr. K. Iyappan

Fourier series.

Continous function:

A function $f(x)$ is said to be continous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Example: $f(x) = x^2, \quad 0 \leq x \leq 1$

Open intervel.

(a, b) takes all the values inbetween a and b

This intervel does not take the values a and b Hence

a and b are discontinous point.

Closed intervel: $[a, b]$ takes all the values from

a to b. Hence all the points are continous.

Left limit:

$$f(a-) = \lim_{h \to 0} f(a-h)$$

Right limit:

$$f(a+) = \lim_{h \to 0} f(a+h)$$

Periodic function:

A function $f(x)$ is said to be periodic if

$f(x+p) = f(x) \ \forall x$ & some values of p

This smallest value of p is called period

of the function $f(x)$

Example : $f(x) = \sin x$.

Hence, $\sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi) = \cdots$

Here the smallest value 2π is the period of $f(x)$

ODD function:

A function $f(x)$ defined in symmetric interval is said to be odd if $f(-x) = -f(x) \ \forall x$

Example

$$f(x) = \sin x$$

$$f(-x) = \sin(-x)$$

$$= -\sin x$$

$$= -f(x)$$

Hence $\sin x$ is an odd function.

$x, x^3, x^5, \ldots$ $x+x^3$

Even function.

A function $f(x)$ defined in symmetric interval is said to be even if $f(-x) = f(x)$

Example : $f(x) = \cos x$

$$f(-x) = \cos(-x)$$

$$= \cos x$$

$$= f(x)$$

Hence $\cos x$ is an even function.

$|x|, x^2, x^4, \ldots$ k

Neither even nor odd

A function $f(x)$ is not satisfying the conditions $f(-x) = -f(x)$ and $f(-x) = f(x)$ is called Neither even nor odd function.

Example.

$$f(x) = 1+x+x^2$$

$$f(-x) = 1-x+x^2$$

$$\neq f(x)$$

$$\neq -f(x)$$

Bernoulli's Formula:

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$\int u\,dv = uv - \int v\,du.$$

sin & cosine values

$\sin\pi = 0$

$\sin 2\pi = 0$

$\sin 3\pi = 0$

$\vdots$

$\sin n\pi = 0$

$\sin(2n-1)\pi = 0$

$\sin(2n+1)\pi = 0$

$\sin 2n\pi = 0$

$\sin(n-1)\pi = 0$

$\sin(n+1)\pi = 0$

$\sin(n+1)\pi/2 = \cos n\pi/2$

$\sin(n-1)\pi/2 = -\cos n\pi/2$

$\sin \pi/2 = 1$ $\qquad \sin 9\pi/2 = 1$

$\sin 3\pi/2 = -1$ $\qquad \sin 11\pi/2 = -1$

$\sin 5\pi/2 = 1$

$\sin 7\pi/2 = -1$

$\cos 0 = 1$

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$\cos 4\pi = 1$

$\vdots$

$\cos n\pi = (-1)^n$

$\cos(2n-1)\pi = -1$

$\cos(2n+1)\pi = -1$

$\cos 2n\pi = 1$

$\cos(n-1)\pi = (-1)^{n-1}$

$\cos(n+1)\pi = (-1)^{n+1}$

$\cos(n+1/2)\pi = 0$

$\cos(n-1/2)\pi = 0$

$\cos \pi/2 = 0$

$\cos 3\pi/2 = 0$

$\cos 5\pi/2 = 0$

$\cos(2n-1)\pi/2 = 0, \quad n = 1, 2 \ldots$

$\cos(2n+1)\pi/2 = 0, \quad n = 0, 1, \ldots$

DRICHILET'S CONDITION

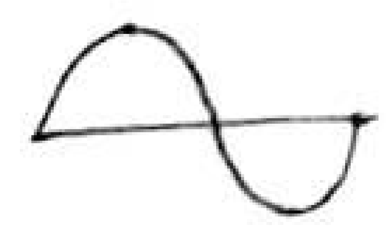
A function $f(x)$ is defined in $(c, c+2l)$ satisfies the following conditions.

(i) $f(x)$ is single valued & finite in $(c, c+2l)$

(ii) $f(x)$ is continous (or) piecewise continuous with finite number of discontinuities in $(c, c+2l)$

(iii) $f(x)$ has no maximum or minimum (or) finite number of maximum or minimum in $(c, c+2l)$

(iv) $f(x)$ is periodic and bounded in $(c, c+2l)$



FOURIER SERIES

A function $f(x)$ defined in $(c, c+2l)$ can be expressed in the form of infinite trignometric series

$$\boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$ is called

Fourier series.

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

Hence. a_0, a_n & b_n are called Euler's formula (or) fourier co-efficients (or) fourier constant.

CONVERGENCE OF FOURIER SERIES

(i) If $x = a$ is a continous point in $(c, c+2l)$ then the fourier series convergence to $f(a)$

i.e, $\left[\text{The value of fourier series}\right]_{x=a} = f(a)$

(ii) If $x = c$ (or) $c+2l$ is a discontinous endpoint then the fourier series convergence to $\frac{f(c) + f(c+2l)}{2}$

i.e., $\left[\text{The value of fourier series}\right]_{c,\, c+2l}$

$$= \frac{f(c) + f(c+2l)}{2}$$

(iii) If $x = a$ is a discontinous midle point. then the fourier series convergence to $\frac{f(a-) + f(a+)}{2}$.

i.e., $\left[\text{The value of fourier series}\right]_{x=a}$

$$= \frac{f[a-] + f[a+]}{2}$$

Expand $f(x) = \begin{cases} x, & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$ as a Fourier Series and hence deduced that $1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$.

Given

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{--- (1)}.$$

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$$

$$a_0 = 1/\pi \int_0^{2\pi} f(x)\, dx$$

$$= 1/\pi \left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right]$$

$$= 1/\pi \left\{ \left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi} \right\}.$$

$$= 1/\pi \left\{ \left[\frac{\pi^2}{2}\right] + \left[2\pi^2 - 3/2\, \pi^2\right] \right\}$$

$$= 1/\pi \left[2\pi^2 - \pi^2\right]$$

$$= 1/\cancel{\pi} \times \pi^{\cancel{2}}$$

$$\boxed{a_0 = \pi}$$

$$a_n = 1/\pi \int_0^{2\pi} f(x) \cos nx\, dx.$$

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$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right.$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] + \left[-\frac{1}{n^2} - \left(-\frac{(-1)^n}{n^2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \frac{-4}{\pi n^2}}$$ if n is odd.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + (2\pi - x)\left(\frac{-\cos nx}{n}\right.$$

$$\left. - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

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$= \frac{1}{\pi}\left\{\left[-\pi(-1)^n/n\right] + \left[\pi/n(-1)^n\right]\right\}$

$\boxed{b_n = 0}$

Sub the values of a_0, a_n & b_n in equation ①.

$$f(x) = \pi/2 - 4/\pi \sum_{n=1,3,5\ldots}^{\infty} 1/n^2 \cos nx \quad \text{——— ②}$$

Put $x=0$ in ②

which is discontinous end point

$$\pi/2 - 4/\pi \sum_{n=1,3,5,\ldots}^{\infty} 1/n^2 = \frac{f(0) + f(2\pi)}{2}$$

$$= \frac{0+0}{2}$$

$$= 0$$

$$\cancel{-} \, 4/\pi \sum_{n=1,3,5}^{\infty} 1/n^2 = \cancel{-}\,\pi/2$$

$$\sum_{n=1,3,5}^{\infty} 1/n^2 = \pi/2 \cdot \pi/4$$

$$= \pi^2/8$$

$$\sum_{n=1,3,5}^{\infty} 1/n^2 = \pi^2/8 \text{ (Ans)}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Fourier Series of the function $f(x)$ in the form of $(-\pi, \pi)$

Find the Fourier series for the function $f(x) = 1+x+x^2$ in $(-\pi, \pi)$ and hence find the value of series $1/1^2 + 1/2^2 + 1/3^2 + \cdots$

Given

$$f(x) = 1+x+x^2$$

$$f(-x) = 1-x+x^2$$

$$f(-x) \neq +f(x)$$

$$f(-x) \neq -f(+x)$$ Hence $f(x)$ is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---(1).}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x+x^2)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x^2)\,dx \quad \left[\because \int_{-\pi}^{\pi} x\,dx = 0\right.$$

because x is an odd function

$$= \frac{2}{\pi}\int_{0}^{\pi} (1+x^2)\,dx.$$

$$= \frac{2}{\pi}\left[x + \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi + \pi^3/3\right]$$

$$= \frac{2}{\pi}\,\pi\left[1 + \pi^2/3\right]$$

$$a_0 = 2\left(1+\pi^2/3\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (1+x^2) \cos nx \, dx$$

$\int_{-\pi}^{\pi} x \cos nx \, dx = 0$ because it is odd

$$= \frac{2}{\pi} \int_{0}^{\pi} (1+x^2) \cos nx \, dx.$$

$$= \frac{2}{\pi} \left[(1+x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right) \right]$$

$$= \frac{2}{\pi} \left[\frac{2\pi (-1)^n}{n^2} \right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx$$

$\int_{-\pi}^{\pi} (1+x^2) \sin nx \, dx = 0$ because it is odd

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\pi (-1)^n}{n} \right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

sub the values of a_0, a_n & b_n in ①

$$f(x) = 1 + \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Put $x = \pi$ in (2).

Which is discontinous end point.

$$1 + \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{f(\pi) + f(-\pi)}{2}$$

$$= \frac{1 + \pi + \pi^2 + 1 - \pi + \pi^2}{2}$$

$$= 1 + \pi^2$$

$$\cancel{1} + \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^{2n}/n^2 = \cancel{1} + \pi^2$$

$$4 \sum_{n=1}^{\infty} (-1)^{2n}/n^2 = \pi^2 - \pi^2/3$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{2\pi^2}{3 \times 4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{\pi^2}{6}$$

$$1/1^2 + 1/2^2 + 1/3^2 + \cdots = \pi^2/6 \text{ (Ans)}$$

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$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

Hence show that $\sum_{n=1}^{\infty} (2n-1)^{-2} = \pi^2/8$.

<u>Given</u>

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi} & 0 \le x \le \pi \\ 1 + 2x/\pi & -\pi \le x \le 0 \end{cases}$$

$$f(-x) = f(x)$$

Hence $f(x)$ is even

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\,dx$$

$$= 2/\pi \int_0^{\pi} (1 - 2x/\pi)\,dx.$$

$$= \frac{2}{\pi}\left[x - \frac{\cancel{2}x^2}{\pi}\right]_0^{\pi}$$

$$a_0 = 0$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\,dx.$$

$$= 2/\pi \int_0^{\pi} (1 - 2x/\pi) \cos nx\,dx$$

$$= 2/\pi \left[(1 - 2x/\pi)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= 2/\pi \left[\frac{-2}{\pi}\frac{(-1)^n}{n^2} + \frac{2}{\pi n^2}\right]$$

$$= 4/\pi^2 n^2 \left[1 - (-1)^n\right]$$

$$a_n = \frac{8}{\pi^2 n^2} \text{ when } n \text{ is odd.}$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] \quad \text{--- (2)}$$

Put $x = 0$ in (2) which is continuous point

$$\frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = f(0)$$
$$= 1$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

ourier Series of the function $f(x)$ in the form of $(0, \pi)$

Express $f(x) = x(\pi - x)$ $0 < x < \pi$ as a Fourier series of Periodicity 2π, containing cosine terms only, hence deduce (H.f.C). $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Soln

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx.$$

$$= \frac{2}{\pi}\left[\frac{x^2\pi}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\cos nx\,dx.$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(\frac{\sin nx}{n}\right) - (\cancel{\pi} - 2x)\left(-\frac{\cos nx}{n^2}\right) + (\cancel{1}-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(-\pi)\frac{(-1)^n}{n^2} + \pi/n^2\right]$$

$$= 2/\pi \left[\frac{-\pi(-1)^n}{n^2} + \pi/n^2\right] = \frac{2}{\cancel{\pi}}\frac{-\cancel{\pi}}{n^2}\left[(-1)^n + 1\right]$$

$$= \frac{-2}{n^2}\left[(-1)^n + 1\right.$$

$$= \frac{-4}{n^2} \text{ if } n \text{ is even.}$$

Sub the values of a_0, a_n in ①.

$$f(x) = \frac{\pi^2}{6} - 4\sum_{n=2,4,\ldots}^{\infty} 1/n^2 \cos nx.$$

Put $x = \pi/2$

which is continous point.

$$\frac{\pi^2}{6} - 4\sum_{n=2,4}^{\infty} 1/(n)^2 \cos n\pi/2 = \pi^2/4$$

$$\cancel{\sum_{n=1}^{\infty} 1/n^2 \cos 2nx = \pi^2/4 - \pi^2/6}$$

$$-4\sum_{n=2,4\ldots} \frac{1}{n^2}\cos n\pi/2 = \frac{\pi^2}{4} - \frac{\pi^2}{6}$$

$$= \frac{2\pi^2}{24}$$

$$\cancel{\sum_{n=1}^{\infty} 1/n^2 \cos 2nx = \pi^2/12}$$

(Ans)

$$-4\left[-\frac{1}{2^2} + \frac{1}{4^2} - 1/6^2 - \cdots\right] = \frac{2\pi^2}{24}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{48}$$

$$\frac{1}{4}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \pi^2/48 \ldots$$

Find the Fourier series of $f(x) = \sin x \quad -\pi < x < \pi$

Given

$f(x) = \sin x$

$f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$

Hence $f(x)$ is odd.

$a_0 = 0, \quad a_n = 0$

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ —— (1)

$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$

$= 2/\pi \int_0^{\pi} \sin x \sin nx$

$= 2/\pi \int_0^{\pi} \sin nx \sin x \, dx.$

$= \frac{\cancel{2}}{\cancel{2}\pi} \int_0^{\pi} \cos(n-1)x - \cos(n+1)x \, dx.$

$= 1/\pi \left[\frac{\sin(n-1)x}{(n-1)}\right]_0^{\pi} - \left[\frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$

$= 1/\pi \times 0$

$\boxed{b_n = 0}$

$b_1 = 2/\pi \int_0^{\pi} f(x) \sin x \, dx.$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \ \sin x \, dx$$

$$= 2/\pi \int_0^{\pi} \sin^2 x \, dx$$

$$= 2/\pi \int_0^{\pi} \frac{1 - \cos 2x}{2}$$

$$= \frac{2}{2\pi} \int_0^{\pi} (1 - \cos 2x) \, dx.$$

$$= 1/\pi \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= 1/\pi \times \pi$$

$$\boxed{b_1 = 1}$$

Sub the values of b_n $ b_1 values in ①.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$f(x) = 1 \times \sin x$$

$$f(x) = \sin x$$

Hence proved.

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Half range cosine series:-

Find the half range cosine series for $f(x) = x\sin x$ in interval $(0, \pi)$ and hence deduce that $1 + 2/1.3 - 2/3.5 + 2/5.7 - \dots = \pi/2$

Given

$f(x) = x \sin x$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi} x \sin x\, dx$$

$$= \frac{2}{\pi} \left[x(-\cos x) - (-\sin x) \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}} \times \cancel{\pi} = 2$$

$$\boxed{a_0 = 2}$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx.$$

$$= 2/\pi \int_0^{\pi} x \cos nx \sin x\, dx$$

$$= 2/2\pi \int_0^{\pi} x \left[\sin(n+1)x - \sin(n-1)x \right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx - \int_0^{\pi} x\sin(n-1)x$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} - \left[x\left(\frac{-\cos(n-1)x}{n-1}\right) + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{n+1}}{n+1} + \cancel{\frac{1}{n+1}} + \frac{\pi(-1)^{n-1}}{n-1} - \cancel{\frac{1}{n-1}}\right]$$

$$= \frac{-2\pi}{\pi\, n^2-1}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] = \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n+1} - \frac{\pi(-1)^n}{n-1}\right]$$

$$= \frac{-2}{n^2-1}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}\right] = \frac{1}{\cancel{\pi}}\cancel{\pi}(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{-2(-1)^n}{n^2-1}$$

$$\boxed{a_n = \frac{-2(-1)^n}{n^2-1}}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} x\sin 2x\,dx$$

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$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

Sub the values in $f(x)$

$$f(x) = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx \quad \text{--- (2)}$$

put $x = \pi/2$ in (2).

which is continous point

$$1 - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos n\pi/2 = f(\pi/2)$$

$$1 - 2\left[-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\right] = \pi/2$$

$$1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots = \pi/2 \text{ (Ans)}$$

Apply the half range cosine Series $f(x) = x$ in $(0, \pi)$

Given

$f(x) = x$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\,dx.$$

$$= 2/\pi \int_0^{\pi} x\,dx$$

$$= 2/\pi \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^2}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\,dx$$

$$= 2/\pi \int_0^{\pi} x \cos nx\,dx.$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos n.x}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - 1/n^2\right]$$

$$= 2/\pi n^2 \left[(-1)^n - 1\right]$$

$$n = \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd.}$$

Sub a_0 & a_n in (1)

$$f(x) = \pi/2 - 4/\pi \sum_{n=1,3,5\cdots}^{\infty} 1/n^2 \cos nx. \quad \text{(Ans)}$$

Half range Sine Series :

Find the half range Sine Series of $f(x) = x\cos x$ in $(0,\pi)$

Given $f(x) = x\cos x$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \cos x \, dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \int_0^{\pi} x \sin(n+1)x + \sin(n-1)x \, dx.$$

$$= 1/\pi \int_0^{\pi} x \sin(n+1)x \, dx + \int_0^{\pi} x \sin(n-1)x \, dx$$

$$= 1/\pi \left[x \frac{-\cos(n+1)x}{n+1} - (1) \frac{-\sin(n+1)x}{(n+1)^2} \right]_0^{\pi} + \left[x \frac{-\cos(n-1)x}{n-1} - (1) \frac{-\sin(n-1)x}{(n-1)^2} \right]_0^{\pi}$$

$$= 1/\cancel{\pi} \, {-\cancel{\pi}} \left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \right]$$

$$= \left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \right] = (-1)^n \left[\frac{n-1+n+1}{n^2-1} \right]$$

$$b_n = \frac{2n(-1)^n}{n^2-1}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x \, dx$$

$$= 2/\pi \int_0^{\pi} x \cos x \sin x \, dx$$

$$= \cancel{2}/\cancel{2}\pi \int_0^{\pi} x \sin 2x \, dx$$

$$= 1/\pi \left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= 1/\pi \left[-\pi/2\right]$$

$$\boxed{b_1 = -1/2}$$

Sub the values b_n \$ b_1 in equ ①

$$f(x) = -1/2 \sin x + 2\sum_{n=2}^{\infty} \frac{n}{n^2-1}(-1)^n \sin nx$$

Fourier Series of the function f(x) in the form of $(0, 2l)$

Abtain the Fourier series for

$$f(x) = \begin{cases} l - x & 0 < x < l \\ 0 & l < x < 2l \end{cases}$$ hence reduce that

(i) $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \pi/4$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8$.

Given

$$f(x) = \begin{cases} l - x, & 0 < x < l \\ 0, & l < x < 2l. \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$$

$$= \frac{1}{l}\int_0^{l} (l - x)\, dx$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{l^2}{2}\right]$$

$$a_0 = l/2$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\, dx$$

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$$= 1/l \int_0^l (l-x) \cos \frac{n\pi x}{l} \, dx$$

$$= 1/l \left[(l-x) \left(\frac{\sin \frac{n\pi x}{l}}{n\pi/l} \right) - (-1) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= 1/l \left[\frac{-l^2}{n^2\pi^2} (-1)^n + \frac{l^2}{n^2\pi^2} \right]$$

$$= 1/l \left[\frac{l^2}{n^2\pi^2} \right] \left[1 - (-1)^n \right]$$

$$a_n = \frac{2l}{n^2\pi^2} \quad \text{if } n \text{ is Odd}$$

$$b_n = 1/l \int_0^{2l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

$$= 1/l \int_0^l (l-x) \sin \frac{n\pi x}{l} \, dx$$

$$= 1/l \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2} \right) \right]_0^l.$$

$$= 1/l \left[\frac{l^2}{n\pi} \right]$$

$$b_n = l/n\pi$$

$$f(x) = l/4 + \frac{2l}{\pi^2} \sum_{n=1,3\ldots}^{\infty} 1/n^2 \cos \frac{n\pi x}{l} + l/\pi \sum_{n=1}^{\infty} 1/n \sin n\pi x/l.$$

Put $x = l/2$ in ②

which is continuous

$$l/4 + l/\pi \sum_{n=1}^{\infty} 1/n \sin n\pi/2 = f(l/2)$$

$$l/4 + l/\pi \left[1 - 1/3 + 1/5 - \cdots \right] = l/2$$

$$l/\pi \left[1 - 1/3 + 1/5 - \cdots \right] = l/4$$

$$\boxed{1 - 1/3 + 1/5 - \cdots = \pi/4}$$

(ii) put $x = 0$ in ②

which is discontinous end point

$$l/4 + \frac{8l}{\pi^2} \sum_{n=1,3\cdots}^{\infty} 1/n^2 = \frac{f(0) + f(2l)}{2}$$

$$= \frac{l+0}{2}$$

$$\frac{8l}{\pi^2} \left[1/1^2 + 1/3^2 + 1/5^2 + \cdots \right] = l/4$$

$$1/1^2 + 1/3^2 + 1/5^2 + \cdots = \frac{\pi^2}{8}$$

Find the Fourier Series expantion of periodic function $f(x)$ of $2l$ defined by $f(x)$:

$\begin{cases} l+x & ; \ -l \le x \le 0 \\ l-x & ; \ 0 \le x \le l \end{cases}$ and hence reduced that

$$\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$$

<u>Given</u>

$$f(x) = \begin{cases} l+x & , \ -l \le x \le 0 \\ l-x & ; \ 0 \le x \le l. \end{cases}$$

$$f(-x) = \begin{cases} l-x & , \ -l \le -x \le 0 \\ l+x & , \ 0 \le -x \le l. \end{cases}$$

$$= \begin{cases} l-x & ; \ 0 \le x \le l \\ l+x & ; \ -l \le x \le 0 \end{cases}$$

$$f(-x) = f(x)$$

Hence $f(x)$ is even.

$\therefore b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{—— (1)}$$

$$a_0 = 2/l \int_0^l f(x)\, dx.$$

$$= 2/l \int_0^l (l-x)\, dx$$

$$= 2/l \left[lx - \frac{x^2}{2} \right]_0^l$$

$$\boxed{a_0 = l}$$

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$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l^2}{n^2\pi^2}(-1)^n + \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{2}{\cancel{l}}\frac{l^{\cancel{2}}}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$a_n = \frac{4l}{n^2\pi^2} \quad \text{if } n \text{ is odd.}$$

Sub the values a_0 & a_n in ①.

$$f(x) = l/2 + \frac{4l}{\pi^2}\sum_{n=1,3\cdots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \quad \text{——— ②}$$

Put $x = 0$ in ②

which is continous point

$$l/2 + \frac{4l}{\pi^2}\sum_{n=1,3\cdots}^{\infty} 1/n^2 = l$$

$$\frac{4l}{\pi^2}\sum_{n=1,3\cdots}^{\infty} 1/n^2 = l/2$$

$$\sum_{n=1,3\cdots}^{\infty} 1/n^2 = \pi^2/8$$

$$\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$$

Parseval's Indentity:

@ If $f(x)$ is defined in $(-\pi, \pi)$ and which is neither even nor odd the parseval's Identity is given by

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

2. If $f(x)$ is defined in $(0, 2\pi)$ then parseval's Identity

$$\frac{1}{\pi}\int_{0}^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

3. If $f(x)$ is even in $(-\pi, \pi)$ ~~or~~ which is defined for half range cosine series in $(0, \pi)$ then the Parseval's Identity

$$\frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

4. If $f(x)$ is odd in $(-\pi, \pi)$ ~~or~~ which is defined for half range sine in $(0, \pi)$ series then the Parseval's Identity

$$\frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

1. If $f(x)$ is defined in $(-l, l)$ and which is neither even nor odd then the parseval's Identity is given

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

2. If $f(x)$ is defined in $(0, 2l)$ then parseval's Idnity is

$$\frac{1}{l}\int_{0}^{2l}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

3. If $f(x)$ is even is $(-l, l)$ are which is defined series in $(0, l)$ then the for half range cosine parseval's Idnity is

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

4. If $f(x)$ is odd is $(-l, l)$ are which is defined in $(0, l)$ then the for half range sine series parseval's Idnitiy is

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2.$$

Parseval's Identity

Obtain the Fourier Series for

$f(x) = \begin{cases} 1 & \text{in } 0, \pi \\ 2 & \text{in } \pi, 2\pi \end{cases}$ and hence find the value of the series $1/1^2 + 1/3^2 + 1/5^2 + \dots$

Given

$$f(x) = \begin{cases} 1 & \text{in } 0, \pi \\ 2 & \text{in } \pi, 2\pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 1/\pi \int_0^{2\pi} f(x)\, dx$$

$$= 1/\pi \int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 2\, dx$$

$$= 1/\pi \left[x\right]_0^{\pi} + \left[2x\right]_{\pi}^{2\pi}$$

$$= 1/\pi \left[\pi + 4\pi - 2\pi\right]$$

$$= 1/\pi \left[\pi + 2\pi\right]$$

$$= 1/\cancel{\pi} \times 3\cancel{\pi}$$

$$\boxed{a_0 = 3}$$

$$a_n = 1/\pi \int_0^{2\pi} f(x) \cos nx\, dx.$$

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$$= \frac{1}{\pi} \int_0^{\pi} 1 \cos nx + \int_{\pi}^{2\pi} 2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left[1 \frac{\sin nx}{n} - 0 \right]_0^{\pi} + \left[2\left(\frac{\sin nx}{n}\right) - 0 \right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi} \times 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} 1 \sin nx \, dx + \int_{\pi}^{2\pi} 2 \sin nx \, dx$$

$$= \frac{1}{\pi} \left[(1)\left(\frac{-\cos nx}{n}\right) - 0 \right]_0^{\pi} + \left[2\left(-\frac{\cos nx}{n}\right) - 0 \right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{-(-1)^n}{n} + \frac{1}{n} \right] + \left[\frac{-2}{n} + \frac{2(-1)^n}{n} \right]$$

$$= \frac{1}{\pi} \left[\frac{+(-1)^n}{n} - \frac{1}{n} \right]$$

$$b_n = \frac{-2}{n\pi} \text{ if } n \text{ is odd}$$

Sub the values of a_0 a_n & b_n in ①

$$f(x) = \frac{3}{2} - \frac{2}{\pi} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n} \sin nx$$

using parseval's Identity

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{1}{\pi}\left[\int_0^{\pi} dx + \int_{\pi}^{2\pi} 4\,dx\right] = 9/2 + \sum_{n=1,3\cdots}^{\infty} 4/\pi^2 \; 1/n^2$$

$$\frac{1}{\pi}\left[[x]_0^{\pi} + 4[x]_{\pi}^{2\pi}\right] = 9/2 + 4/\pi^2 \sum_{n=1,3\cdots}^{\infty} 1/n^2$$

$$\frac{1}{\pi}\left[\pi + 4\pi\right] = 9/2 + 4/\pi^2 \sum_{n=1,3\cdots}^{\infty} 1/n^2$$

$$\frac{1}{\pi} \times 5\pi = 9/2 + 4/\pi^2 \sum_{n=1,3\cdots}^{\infty} 1/n^2$$

$$1/2 = 4/\pi^2 \sum_{n=1,3\cdots}^{\infty} 1/n^2$$

$$\sum_{n=1,3\cdots}^{\infty} 1/n^2 = \pi^2/8$$

$$1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8 \quad \text{(Ans)}$$

Obtain the Fourier series for $f(x) = x^2$ $(-\pi, \pi)$ and hence reduced that.

(i) $1/1^4 + 1/2^4 + 1/3^4 + \cdots$

(ii) $1/1^2 + 1/2^2 + 1/3^2 + \cdots$

(iii) $1/1^2 - 1/2^2 + 1/3^2 + \cdots$

(iv) $1/1^2 + 1/3^2 + 1/5^2 + \cdots$

Given

$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$ Hence $f(x)$ is even

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ ------ (1)}$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi} x^2\, dx$$

$$= 2/\pi \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx.$$

$$= 2/\pi \int_0^{\pi} x^2 \cos nx\, dx$$

$$= 2/\pi \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(-\frac{\cos nx}{n^2} \right) + (-2) \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= 2/\pi \left[2\pi \frac{(-1)^n}{n^2} - 0 \right]$$

$$= \frac{4\cancel{\pi}}{\cancel{\pi}} (-1)^n / n^2$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} 1/n^2 (-1)^n \cos nx \quad \text{—— ②}$$

$$2/\pi \int_0^\pi [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$2/\pi \int_0^\pi x^4 dx = 1/\cancel{2} \frac{\overset{2}{\cancel{4}}\pi^4}{9} + \sum_{n=1}^{\infty} 16/n^4 (-1)^{2n}$$

$$2/\pi \left[\frac{x^5}{5}\right]_0^\pi = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^4}$$

$$2/\cancel{\pi} \times \frac{\pi^{\cancel{5}}}{5} = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^4}$$

$$2\pi^4 (1/5 - 1/9) = 16 \sum_{n=1}^{\infty} (-1)^{2n}/n^4$$

$$\sum_{n=1}^{\infty} (-1)^{2n}/n^4 = \pi^4/90$$

$$1/1^4 + 1/2^4 + 1/3^4 + \dots = \pi^4/90$$

(i) To find $1/1^2 + 1/2^2 + 1/3^2 + \dots\dots$

put $x = \pi$ in ②.

Which is discontinous end point.

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \cancel{x} = \frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2 + \pi^2}{2}$$

$$= \frac{\cancel{2}\pi^2}{\cancel{2}} = \pi^2$$

$$4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \pi^2 - \pi^2/3$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{\cancel{2}\pi^2}{\cancel{12}\,6}$$

$$1/1^2 + 1/2^2 + 1/3^2 + \cdots = \pi^2/6 \quad \text{--- (3)}$$

(iii) To find
$$1/1^2 - 1/2^2 + 1/3^2 + \cdots \cancel{= \pi^2/12}$$

put $x=0$ in (2)

which is continous point

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = f(0)$$

$$\frac{\pi^2}{3} + 4\left[-1/1^2 + 1/2^2 + 1/3^2 + \cdots\right] = 0$$

$$1/1^2 - 1/2^2 + 1/3^2 - \cdots = \pi^2/12 \quad \text{--- (4)}$$

(iv) $1/1^2 + 1/3^2 + 1/5^2 + \cdots$

$$(3) + (4) \Rightarrow \quad 2\, 1/1^2 + 2\, 1/3^2 + 2\, 1/5^2 + \cdots = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left[1/1^2 + 1/3^2 + 1/5^2 + \cdots\right] = \frac{3\pi^2}{12}$$

$$1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$$

Obtain the cosine series for the function $f(x) = x$ in $(0, \pi)$ hence reduced that $\sum_{n=1}^{\infty} 1/(2n-1)^4 = \pi^4/96$

Given

$f(x) = x.$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——} ①.$$

$$a_0 = 2/\pi \int_0^{\pi} x \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^2}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx \, dx$$

$$= 2/\pi \int_0^{\pi} x \cos nx \, dx$$

$$= 2/\pi \left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - 1/n^2\right]$$

$$= 2/n^2\pi \left[(-1)^n - 1\right]$$

$$a_n = \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd.}$$

Sub the a_0 & a_n values in eq ①.

$$f(x) = \pi/2 - 4/\pi \sum_{n=1,3,5\ldots}^{\infty} 1/n^2 \cos nx.$$

Using Parseval's Identity

$$2/\pi \int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$2/\pi \int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + 16/\pi^2 \sum_{n=1,3\ldots}^{\infty} 1/n^4$$

$$2/\pi \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3.}^{\infty} 1/n^4$$

$$2/\pi \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} 1/(2n-1)^4$$

$$2/\pi \frac{\pi^3}{3} = \frac{\pi^2}{2} + 16/\pi^2 \sum_{n=1}^{\infty} 1/(2n-1)^4$$

$$\pi^2 (2/3 - 1/2) = 16/\pi^2 \sum_{n=1}^{\infty} 1/(2n-1)^4$$

$$\frac{\pi^2}{6} = 16/\pi^2 \sum_{n=1}^{\infty} 1/(2n-1)^4$$

$$\sum_{n=1}^{\infty} 1/(2n-1)^4 = \pi^4/96$$

$$1/1^4 + 1/3^4 + 1/5^4 + \ldots = \pi^2/96 \text{ (Ans)}$$

Abtain the half range sine Series expantion of period l for the function $f(x) = \begin{cases} x & (0, l/2) \\ l-x & (l/2, l) \end{cases}$

and deduced the sum of $\sum_{n=1}^{\infty} 1/(2n-1)^4 = \pi^4/96$

Given

$$f(x) = \begin{cases} x & 0, l/2 \\ l-x & l/2, l. \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{——} (1).$$

$$b_n = 2/l \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

$$= 2/l \int_0^{l/2} x \sin \frac{n\pi x}{l} dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l} dx.$$

$$= 2/l \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2} \right) \right]_0^{l/2} + \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2} \right) \right]_{l/2}^{l}$$

$$= 2/l \left[\frac{-l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right] + \left[\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$= \frac{2}{\cancel{l}} \times \frac{2l^{\cancel{2}}}{n^2\pi^2} \sin n\pi/2$

$= \frac{4l}{n^2\pi^2} \sin n\pi/2$ if n is odd.

$f(x) = \frac{4l}{\pi^2} \sum_{n=1,3\cdots}^{\infty} 1/n^2 \sin n\pi/2 \sin n\pi x/l.$

using parseval's Identity

$$2/l \int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} bn^2$$

$$2/l \int_0^{l/2} x^2 dx + \int_{l/2}^{l} (l^2 - 2lx + x^2) dx = \frac{16l^2}{\pi^4} \sum_{n=1,3\cdots}^{\infty} 1/n^4 \; (\sin(n\pi/2))^2.$$

$$2/l \left[\left(\frac{x^3}{3}\right)_0^{l/2} + \left[l^2x - \frac{\cancel{2}lx^2}{\cancel{2}} + \frac{x^3}{3}\right]_{l/2}^{l}\right] = \frac{16l^2}{\pi^4} \sum_{n=1,3\cdots}^{\infty} 1/n^4$$

$$2/l \left\{ l^3/24 + \left[l^3 - l^3 + l^3/3 \right] - \left(l^3/2 - l^3/4 + l^3/24 \right) \right\} = \frac{16l^2}{\pi^4} \sum_{n=1,3\cdots}^{\infty} 1/n^4$$

$$2/l \left[\frac{l^3}{3} - l^3/4 \right] = \frac{16l^2}{\pi^4} \sum_{n=1,3\cdots}^{\infty} 1/n^4$$

$$2/\cancel{l} \; \frac{l^{\cancel{3}}}{12} = \frac{16l^2}{\pi^4} \sum_{n=1,3\cdots}^{\infty} 1/n^4$$

$$\frac{\pi^4}{96} = \sum_{n=1,3\cdots}^{\infty} 1/n^4$$

HARMONIC ANALYSIS:-

The process of finding fourier series by giving a finite number of values of a function at finite number of points is called harmonic analysis.

Formula:

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \dots$$

$$a_0 = \frac{2 \sum y}{n}$$

$$a_1 = 2 \frac{\sum y \cos x}{n}$$

$$b_1 = 2 \frac{\sum y \sin x}{n}$$

$$a_2 = 2 \frac{\sum y \cos 2x}{n}$$

$$b_2 = 2 \frac{\sum y \sin 2x}{n}$$

$$a_3 = 2 \frac{\sum y \cos 3x}{n}$$

$$b_3 = 2 \frac{\sum y \sin 3x}{n}$$

Type I : π form or degree form

Find the fourier series upto third harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table of the values given below.

0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
1	1.4	1.9	1.7	1.5	1.2	1

Given:

$y = f(x) \qquad (0, 2\pi)$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$$

$n = 6$

$$a_0 = \frac{2 \sum y}{n}$$

$$= 2 \times \frac{8.7}{6}$$

$$a_0 = 2.9$$

$$a_1 = \frac{2 \sum y \cos x}{n}$$

$$= 2 \times \frac{-1.1}{6}$$

$$a_1 = -0.3667$$

$$b_1 = \frac{2 \sum y \sin x}{n}$$

$$= 2 \times \frac{0.5196}{n}$$

$$b_1 = 0.1732$$

$$a_2 = \frac{2 \sum y \cos 2x}{n}$$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$	$y\cos 3x$	$y\sin 3x$
0	1	1	0	1	0	1	0	1	0	1	0	1	0
$\pi/3$	1.4	0.5	0.866	-0.5	0.866	-1	0	0.7	1.2124	-0.7	1.2124	-1.4	0
$2\pi/3$	1.9	-0.5	0.866	-0.5	-0.866	1	0	-0.95	1.6454	-0.95	-1.6454	1.9	0
π	1.7	-1	0	1	0	-1	0	-1.7	0	1.7	0	-1.7	0
$4\pi/3$	1.5	-0.5	-0.866	-0.5	0.866	1	0	-0.75	-1.299	-0.75	1.299	1.5	0
$5\pi/3$	1.2	0.5	-0.866	-0.5	-0.866	-1	0	0.6	-1.0392	-0.6	-1.0392	-1.2	0
	$\Sigma y = 8.7$							$\Sigma y\cos x$ $= -1.1$	$\Sigma y\sin x$ $= 0.5196$	$\Sigma y\cos 2x$ $= -0.3$	$\Sigma y\sin 2x$ $= -0.1732$	$\Sigma y\cos 3x$ $= 0.1$	$\Sigma y\sin 3x$. 0

$$= 2 \times \frac{-0.3}{6}$$

$$a_2 = -0.1$$

$$b_2 = \frac{2 \sum y \sin 2x}{6}$$

$$= \frac{-2 \times 0.1732}{6}$$

$$b_2 = -0.0577$$

$$a_3 = 2 \frac{\sum y \cos 3x}{n}$$

$$= 2 \times 0.1/6$$

$$a_3 = 0.03$$

sub the values of $a_0, a_1, a_2, a_3, b_1, b_2, b_3$

$$f(x) = 1.45 + (-0.367 \cos x + 0.1732 \sin x) + (-0.1 \cos 2x - 0.0577 \sin 2x) + (0.03 \cos 3x)$$

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Type 2: T form

The values of x at the corresponding values of $f(x)$ over a period T are given table

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.3	-0.88	-0.25	1.98.

show that $f(x) = 0.75 + 0.37\cos\theta + 1.004 \sin\theta$ where $\theta = 2\pi x/T$

Soln

$$f(x) = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) \quad \text{—— (1).}$$

$$a_0 = 2\frac{\Sigma y}{n}$$

$$= 2 \times \frac{4.5}{6}$$

$$a_0 = 1.5$$

$$a_1 = 2\frac{\Sigma y\cos\theta}{n}$$

$$= 2 \times \frac{1.12}{6}$$

$$a_1 = 0.37$$

$$b_1 = 2\frac{\Sigma y \sin\theta}{n}$$

$$= 2 \times \frac{3.0136}{6}$$

$$b_1 = 1.004.$$

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x	$\theta = 2\pi x/T$	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1	0.	1.98	0
$T/6$	$\pi/3$	1.30	0.5	0.866	0.65	1.1258
$T/3$	$2\pi/3$	1.05	-0.5	0.866	-0.525	0.9093
$T/2$	π	1.3	-1	0	-1.3	0
$2T/3$	$4\pi/3$	-0.88	-0.5	-0.866	0.44	0.4620
$5T/6$	$5\pi/3$	-0.25	0.5	-0.866	0.125	0.2165
		$\Sigma y =$ 4.5			$\Sigma y\cos\theta =$ 1.18	$\Sigma y\sin\theta$ = 3.0136.

Sub the values in eq ①.

$f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$.

Type 3 : L form:

Find the constant term an coefficient of first sine and cosine terms in the Fourier expansion of y as given in the following table

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln

$n = 6$

$2l = 6$

$l = 3$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \quad \text{——} ①.$$

$$a_0 = 2\frac{\Sigma y}{n}$$

$$= \frac{2 \times 125}{6}$$

$$a_0 = 41.6$$

$$a_1 = 2\frac{\Sigma y \cos \pi x/3}{n}$$

$$a_1 = 2 \times \frac{-25}{6}$$

$$a_1 = -8.33$$

$$b_1 = \frac{2\sum y \sin \pi x/3}{6}$$

$$= 2 \times \frac{3.464}{6}$$

$$b_1 = -1.154.$$

x	y	$\cos \pi x/3$	$\sin \pi x/3$	$y \cos \pi x/3$	$y \sin \pi x/3$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	−0.5	0.866	−12	20.784
3	28	−1	0	−28	0
4	26	−0.5	−0.866	−13	−22.516
5	20	0.5	−0.866	10	−17.32
	$\sum y = 125$			$\sum y \cos \pi x/3 = -25$	$\sum y \sin \pi x/3 = -3.464$

Sub the values in ①.

$f(x) = 20.835 + 8.33 \cos\frac{\pi x}{3} - 1.154 \sin\frac{\pi x}{3}$

Obtain the first three numbers of harmonic

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Soln

$f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos \pi x + b_3 \sin \pi x\right)$

$a_0 = \frac{2 \Sigma y}{n}$

$= 2 \times \frac{42}{6}$

$a_0 = 14$

$a_1 = 2 \frac{\Sigma y \cos \pi x/3}{6}$

$= \frac{-8.5}{3}$

$a_1 = -2.883$

$a_2 = 2 \frac{\Sigma y \cos 2\pi x/3}{n}$

$= 2 \times \frac{-2.5}{6}$

$a_2 = -0.833$

$a_3 = 2 \frac{\Sigma y \cos 3\pi x/3}{n}$

$= 2 \times 8/6$

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x	y	$\cos \pi x/3$	$\sin \pi x/3$	$\cos 2\pi x/3$	$\sin 2\pi x/3$	$\cos 3\pi x/3$	$\sin 3\pi x/3$	$y \cos \pi x/3$	$y \sin \pi x/3$	$y \cos 2\pi x/3$	$y \sin 2\pi x/3$	$y \cos 3\pi x/3$	$y \sin 3\pi x/3$
0	4	1	0	1	0	1	0	4	0	4	0	4	0
1	8	0.5	0.866	-0.5	0.866	-1	0	4	6.928	-4	6.928	-8	0
2	15	-0.5	-0.866	-0.5	-0.866	1	0	-7.5	12.99	-7.5	-12.99	+15	0
3	7	-1	0	+1	0	-1	0	-7	0	7	0	-7	0
4	6	-0.5	-0.866	-0.5	0.866	1	0	-3	-5.196	-8	5.196	6	0
5	8	0.5	-0.866	-0.5	-0.866	-1	0	1	-1.732	1	-1.732	-2	0
	$\Sigma y = 42$							$\Sigma y \cos \pi x/3 = -8.5$	$\Sigma y \sin \pi x/3 = 12.99$	$\Sigma y \cos 2\pi x/3 = -2.5$	$\Sigma y \sin 2\pi x/3 = -2.598$	$\Sigma y \cos 3\pi x/3 = +8$	$\Sigma y \sin$

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$a_3 = 2.667$

$$b_1 = 2\frac{\Sigma y \sin \pi x/3}{n}$$

$$= 2 \times \frac{12.99}{6}$$

$$b_1 = 4.33$$

$$b_2 = 2\frac{\Sigma y \sin 2\pi x/3}{n}$$

$$= 2 \times \frac{-2.598}{6}$$

$$b_2 = -0.866$$

$$b_3 = 0$$

sub the values in ①.

$$f(x) = 7 + (-2.833 \cos \pi x/3 + 4.33 \sin \pi x/3) + (-1.5 \cos 2\pi x/3 - 0.866 \sin 2\pi x/3) + 2.667 \cos \pi x$$

complex form of fourier series

Formula I.

If $f(x)$ is defined in $(-l, l)$ then the complex form of fourier series is defined as $f(x)$

$$= \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

where $C_n = 1/2l \int_{-l}^{l} f(x) e^{-in\pi x} dx$.

Formula II :

If $f(x)$ is defined in $(-\pi, \pi)$ then the complex form of Fourier series is defined as $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

Formula III :

If $f(x)$ is defined in $(0, 2l)$ then the complex form of Fourier Series $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$.

where $C_n = \frac{1}{2l} \int_{0}^{2l} f(x) e^{-in\pi x} dx$.

Formula III :

If $f(x)$ is defined in $(0, 2\pi)$ then the complex form of fourier series defined in $(0, 2\pi)$ is given by $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$.

where $C_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} dx$.

Show that the complex form of fourier Series of the periodic function $f(x) = e^{-x}$ $-1 < x < 1$ & $f(x+2) = f(x)$ find the complex form of fourier Series.

$f(x) = e^{-x} \qquad -1 < x < 1$

$2l = 2$

$l = 1$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) \, e^{-in\pi x} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-x} \, e^{-in\pi x} \, dx.$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi \cancel{x})x}}{-(1+in\pi x)} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-(1+in\pi)} \right]$$

$$= -\frac{1}{2} \left(\frac{1-in\pi}{1^2+n^2\pi^2} \right) \left[e^{-1} (\cos n\pi - i \sin n\pi) - e^{1} (\cos n\pi + i \sin n\pi) \right]$$

$$= -\frac{1}{2} \left(\frac{1-in\pi}{1+n^2\pi^2} \right) \left[\cos n\pi \, (e^{-1} - e) - i \sin n\pi \, (e^{-1} + e) \right]$$

$$= -\frac{1}{2} \left[\frac{1-in\pi}{1+n^2\pi^2} \right] \left[\cos n\pi \, (e^{-1} - e) \right]$$

$$= -\frac{1}{2} \left[\frac{1-in\pi}{1+n^2\pi^2} \right] (-1)^n \, (e^{-1} - e)$$

$$= \frac{1}{2}\left(\frac{1-in\pi}{1+n^2\pi^2}\right)(e^1 - e^{-1})(-1)^n$$

$$= \frac{(-1)^n}{2}\left[\frac{1-in\pi}{1+n^2\pi^2}\right] 2\sinh 1$$

$$c_n = (-1)^n\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\sinh 1$$

$$f(x) = \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1-in\pi}{1+n^2\pi^2}\right)(\sinh 1)\, e^{in\pi x}.$$

Find the complex form of Fourier Series of $f(x) = e^{-x}$, $-1 < x < 1$ and hence prove that

$$1/\sinh 1 = \sum_{n=-\infty}^{\infty} (-1)^n / 1+n^2\pi^2 \, (1-in\pi)$$

$$f(x) = e^{-x}.$$

$$l = 1$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

where $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x}\, dx.$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx.$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi x)x}}{-(1+in\pi x)}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-(1+in\pi)}\right]$$

$$= \frac{1}{2}\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\left[e^{-1}\cos n\pi - i\sin n\pi) - e\ \cos n\pi + i\sin n\pi)\right]$$

$$= \frac{-1}{2}\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\left[\cos n\pi(e^{-1}-e) - i\sin n\pi\ (e^{-1}+e)\right]$$

$$= -\frac{1}{2}\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\left[\cos n\pi\ (e^{-1}-e)\right]$$

$$= \frac{1}{-2}\left[\frac{1-in\pi}{1+n^2\pi^2}\right](-1)^n(e^{-1}-e)$$

$$= \frac{1}{2}\left[\frac{1-in\pi}{1+n^2\pi^2}\right](e^{1}-e^{-1}](-1)^n$$

$$= \frac{(-1)^n}{2}\left[\frac{1-in\pi}{1+n^2\pi^2}\right] 2\sinh 1$$

$$c_n = (-1)^n\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\sinh 1$$

Sub the c_n in equ ①.

$$f(x) = \sum_{n=-\infty}^{\infty} (-1)^n\left(\frac{1-in\pi}{1+n^2\pi^2}\right)\sinh 1\ e^{in\pi x} \quad \text{—— ②}$$

put $x = 0$ in eq (27).

$$1 = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1$$

$$1/\sinh 1 = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2}$$

Hence proved.

UNIT III

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

ONE DIMENTIONAL WAVE EQUATION

Write the PDE Governing one dimentional wave equation?

The one Dimensional wave equation is

$$y_{tt} = \alpha^2 y_{xx}.$$

1. What is the constant α^2 in the wave equation $u_{tt} = \alpha^2 u_{xx}$ or in the wave equation $\frac{\partial^2 y}{\partial t^2} = \frac{c^2 \partial^2 y}{dx^2}$ what does c^2 stands for.

$$\alpha^2 = T/m = \frac{\text{Tension}}{\text{Mass per unit length of the string}}$$

2. What are the possible solution of 1D wave equation.

(i) $y(x,t) = (A\cos px + B\sin px)(C\cos p\alpha t + D\sin p\alpha t)$

(ii) $y(x,t) = (A_1 x + A_2)(A_3 t + A_4)$

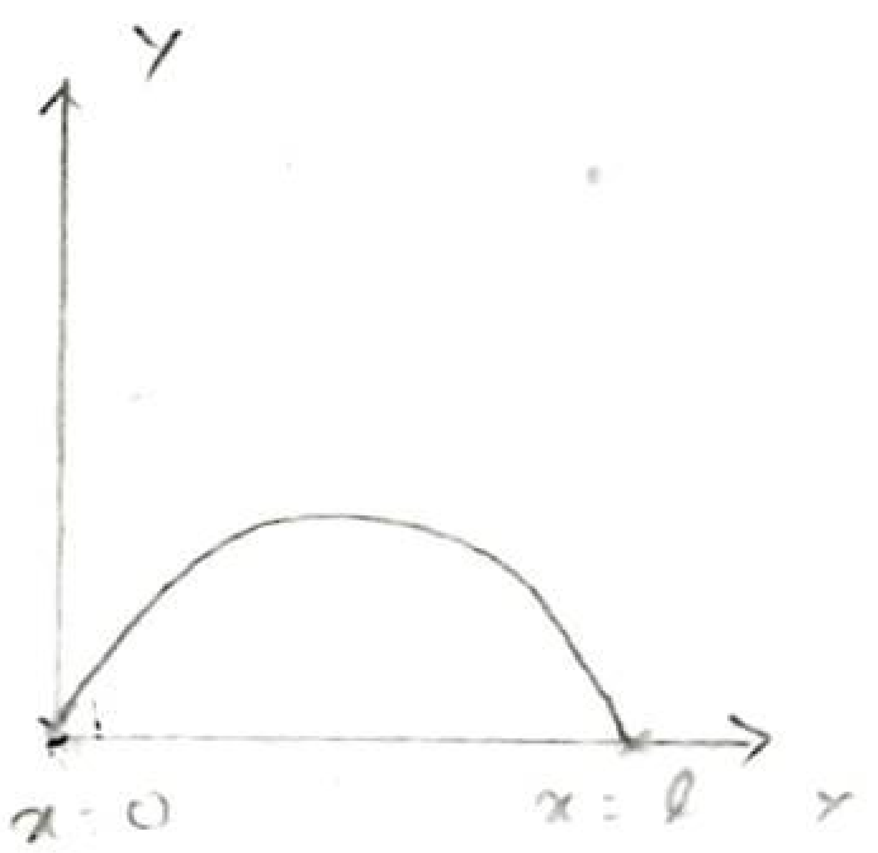
(iii) $y(x,t) = (A_5 e^{tx} + A_6 e^{-tx})(A_7 e^{\alpha t x} + A_8 e^{-\alpha t x})$

3. Write the initial condition of the wave equation if the string has an initial displacement or write the initial condition of the wave equation if the string

STRING WITH ZERO VELOCITY:

1) A String is stretched and fastened to two points $x=0$ and $x=l$ apart. Motion is started by displacement the string into the form $y=k(lx-x^2)$, from which it is released at time $t=0$. Find the Displacement of any point on the string at a distance of x from one end at a time t.

1D Wave eqn is $y_{tt} = \alpha^2 y_{xx}$.



Boundary conditions

(i) $y(0,t) = 0 \quad \forall\ t \geq 0$

(ii) $y(l,t) = 0 \quad \forall\ t \geq 0$

Initial conditions

(iii) $\frac{\partial y}{\partial t}(x,0) = 0 \qquad 0 \leq x \leq l$

(iv) $y(x,0) = k(lx - x^2) \quad 0 \leq x \leq l.$

The suitable solution of 1D wave eqn is

$$y(x,t) = (A\cos p x + B\sin p x)(C\cos p\alpha t + D\sin p\alpha t) \quad \text{---} (1)$$

Apply (i) in (1)

$$A[C\cos p\alpha t + D\sin p\alpha t] = 0$$

$\boxed{A=0}$ $\qquad (\because C\cos p\alpha t + D\sin p\alpha t \neq 0)$

Sub this in ①

$$y(x,t) = B \sin px \left[C\cos p\alpha t + D \sin p\alpha t \right] \text{ --- } ②$$

Apply (ii) in ②

$$B \sin pl \left[C \cos p\alpha t + D \sin p\alpha t \right] = 0 \qquad (\because B \neq 0)$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Sub this in ②

$$y(x,t) = B \sin \frac{n\pi x}{l} \left[C \cos \frac{n\pi \alpha t}{l} + D \sin \frac{n\pi\alpha t}{l} \right] \text{ --- } ③$$

$$\frac{\partial y}{\partial t}(x,t) = B \sin \frac{n\pi x}{l} \left[C \left(-\sin \frac{n\pi\alpha t}{l} \right)\left(\frac{n\pi\alpha}{l} \right) + D\left(\cos \frac{n\pi\alpha t}{l} \right) \left(\frac{n\pi\alpha}{l} \right) \right] \text{ --- } ④$$

Apply (iii) in ④

$$B \sin \frac{n\pi x}{l} \, D \, \frac{n\pi\alpha}{l} = 0$$

$$\boxed{D = 0}$$

Sub this in ③

$$y(x,t) = B \sin \frac{n\pi x}{l} \, C \cos \frac{n\pi\alpha t}{l}.$$

Generalising the above eqn, we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi\alpha t}{l} \text{ --- } ⑤.$$

Apply (iv) in ⑤.

$$\sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{l} = k(lx - x^2)$$

$$B_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx.$$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l.$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n - (-2)\frac{l^3}{n^3\pi^3}\right]$$

$$= \frac{2k}{\cancel{l}} \frac{2l^{\cancel{3}}}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{8kl^2}{n^3\pi^3} \text{ if } n \text{ is odd}$$

sub this value (5)

$$y(x,t) = \sum_{n=1,3\cdots}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi\alpha t}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi\alpha t}{l}.$$

(Ans)

STRING WITH NON ZERO ~~METHOD~~ VELOCITY

1) The string is streched between two fixed point at a distance $2l$ apart and the points of a string are given initial velocities v, where initial velocities $v = \begin{cases} cx/l, & 0 \le x \le l \\ c/l\,(2l-x), & l \le x \le 2l \end{cases}$ x being a distance from one endpoint. Find the displacement of a string at any subsequent time.

1 D wave equation is $y_{tt} = \alpha^2 y_{xx}$

Boundary conditions.

(i) $y(0,t) = 0 \quad \forall\, t \geq 0$

(ii) $y(2l,t) = 0 \quad \forall\, t \geq 0$

Initial conditions.

(iii) $y(x,0) = 0 \quad \forall\, 0 \le x \le 2l.$

(iv) $\dfrac{\partial y}{\partial t}(x,0) = \begin{cases} c/l\; x & \text{in } 0 \le x \le l \\ c/l\,(2l-x) & \text{in } l \le x \le 2l. \end{cases}$

The suitable soln of 1D wave eqn.

$$y(x,t) = (A\cos px + B\sin px)(C\cos p\alpha t + D\sin p\alpha t) \quad \text{--- } ①$$

Apply (i) in ①

$$A\,(C\cos p\alpha t + D\sin p\alpha t) = 0$$

$\boxed{A = 0}$ $\quad (\because C\cos p\alpha t + D\sin p\alpha t \neq 0)$

Sub this in (1)

$$y(x,t) = B \sin px \,(C \cos p\alpha t + D \sin p\alpha t) \quad \text{---} (2)$$

Apply (ii) in (2)

$$B \sin 2lp \,[C \cos p\alpha t + D \sin p\alpha t) = 0$$

$$\sin 2lp = 0$$

$$2lp = n\pi$$

$$p = \frac{n\pi}{2l}$$

Sub this in (2)

$$y(x,t) = B \sin \frac{n\pi x}{2l} \left[C \cos \frac{n\pi\alpha t}{2l} + D \sin \frac{n\pi\alpha t}{2l} \right] \quad \text{---} (3)$$

Apply (iii) in (3)

$$B \sin \frac{n\pi x}{2l} C = 0 \qquad B \neq 0$$

$$\boxed{C = 0}$$

Sub this in (3)

$$y(x,t) = B \sin \frac{n\pi x}{2l} \; D \sin \frac{n\pi\alpha t}{2l}.$$

Generalising the above eqn. we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \sin \frac{n\pi\alpha t}{2l} \quad \text{---} (4).$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi\alpha t}{2l} \left(\frac{n\pi\alpha}{2l}\right) \quad \text{---} (5)$$

Apply (iv) in (5)

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \quad \frac{n\pi\alpha}{2l} = \begin{cases} \frac{c}{l} x & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l. \end{cases}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} = \begin{cases} \frac{c}{l} x \frac{2l}{n\pi\alpha} & \text{in } 0<x<l \\ \frac{c}{l} \frac{2l}{n\pi\alpha}(2l-x) & \text{in } l<x<2l. \end{cases}$$

$$B_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{2l}\, dx$$

$$= \frac{1}{l} \frac{2c}{n\pi\alpha} \left[\int_0^l x \sin \frac{n\pi x}{2l}\, dx + \int_l^{2l} (2l-x) \sin \frac{n\pi x}{2l}\, dx \right]$$

$$= \frac{2c}{n\pi\alpha l} \left\{ \left[x \left(\frac{-\cos \frac{n\pi x}{2l}}{n\pi/2l} \right) - \left(\frac{-\sin \frac{n\pi x}{2l}}{n^2\pi^2/4l^2} \right) \right]_0^l + \left[(2l-x) \left(\frac{-\cos \frac{n\pi x}{2l}}{n\pi/2l} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{2l}}{\frac{n^2\pi^2}{2l}} \right) \right]_l^{2l} \right\}$$

$$= \frac{2c}{n\pi\alpha l} \left\{ \left[\cancel{\frac{-2l^2}{n\pi} \cos \frac{n\pi}{2}} + \frac{4l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right] + \left[\cancel{\frac{2l^2}{n\pi} \cos n\pi/2} + 4l^2/n^2\pi^2 \sin n\pi/2 \right] \right\}$$

$$= \frac{2c}{n\pi\alpha l} \quad \frac{8l^2}{n^2\pi^2} \sin n\pi/2$$

$$B_n = \frac{16cl}{n^3\pi^3\alpha} \sin \frac{n\pi}{2} \text{ if } n \text{ is odd.}$$

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sub this in (4) eqn.

$$y(x,t) = \frac{16cl}{\pi^3 \alpha} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} \sin\frac{n\pi}{2} \sin \frac{n\pi x}{2l} \cdot \sin\frac{n\pi\alpha t}{2l}$$

$$y(x,t) = \frac{16cl}{\pi^3\alpha} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi}{2} \sin\frac{(2n-1)\pi x}{2l}$$

$$\sin\frac{(2n-1)\pi\alpha t}{2l}.$$

2) An Elastic string of length $2l$ fixed at both two ends is distrubed from its equilibrium position by importing to each point and initial velocity of magnitude $k(2lx - x^2)$ Find the displacement $y(x,t)$.

1D wave eqn $y_{tt} = \alpha^2 y_{xx}$.

Boundary conditions.

(i) $y(0,t) = 0 \quad \forall\ t \geq 0$

(ii) $y(2l,t) = 0 \quad \forall\ t \geq 0$

Initial conditions

(iii) $y(x,0) = 0 \quad \forall\ 0 \leq x \leq 2l.$

(iv) $\frac{\partial y}{\partial t}(x,0) = k(2lx - x^2)$

The suitable soln of 1D wave eqn

$$y(x,t) = (A\cos px + B\sin px)(C\cos p\alpha t + D\sin p\alpha t) \quad \text{—— (1).}$$

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Apply (i) in ①

$A\,(C\cos p\alpha t + D\sin p\alpha t) = 0$

$\boxed{A=0}$ $\quad\therefore\ C\cos p\alpha t + D\sin p\alpha t \neq 0$

Sub this in ①

$y(x,t) = B\sin px\,[C\cos p\alpha t + D\sin p\alpha t]$ — ②

Apply (ii) in ②

$B\sin 2lp\,[C\cos p\alpha t + D\sin p\alpha t] = 0$

$\sin 2lp = 0$

$2lp = n\pi$

$\boxed{p = \frac{n\pi}{2l}}$

Sub this ②

$y(x,t) = B\sin\frac{n\pi x}{2l}\left[C\cos\frac{n\pi \alpha t}{2l} + D\sin\frac{n\pi\alpha t}{2l}\right]$ — ③.

Apply (iii) in ③

$C\ B\sin\frac{n\pi x}{2l} = 0$

$\boxed{C=0}$

$y(x,t) = B\sin\frac{n\pi x}{2l}\ D\sin\frac{n\pi\alpha t}{2l}$

Generalising the above eqn.

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi\alpha t}{2l}$ — ④.

$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi\alpha t}{2l}\left(\frac{n\pi\alpha}{2l}\right)$ — ⑤.

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Apply (iv) in (5)

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l} \frac{n\pi\alpha}{2l} = k(2lx - x^2)$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l} = \frac{2lk}{n\pi\alpha}(2lx - x^2)$$

$$B_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\,dx.$$

$$= \frac{2\cancel{l}k}{\cancel{l}n\pi\alpha}\int_0^{2l}(2lx - x^2)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2k}{n\pi\alpha}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi x}{2l}}{n\pi/2l}\right) - (2l - 2x)\left(\frac{-\sin\frac{n\pi x}{2l}}{n^2\pi^2/4l^2}\right) + (-2)\left(\frac{-\cos\frac{n\pi x}{2l}}{n^3\pi^3/8l^3}\right)\right]_0^{2l}$$

$$= \frac{2k}{n\pi\alpha}\left[\frac{-16l^3}{n^3\pi^3}(-1)^n + \frac{16l^3}{n^3\pi^3}\right]$$

$$B_n = \frac{64l^3k}{n^4\pi^4\alpha} \quad \text{if } n \text{ is odd.}$$

sub in (4)

$$y(x,t) = \frac{64l^3k}{\pi^4\alpha}\sum_{n=1,3\ldots}^{\infty} \frac{1}{n^4}\sin\frac{n\pi x}{2l}\sin\frac{n\pi\alpha t}{2l}.$$

$$y(x,t) = \frac{64l^3k}{\pi^4\alpha}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2l}\sin\frac{(2n-1)\pi\alpha t}{2l}.$$

(Ans)

ONE DIMENTIONAL HEAT EQUATION

In steady state condition, derive the solution of one dimensional heat flow equation.

$$u_t = \alpha^2 u_{xx} \quad \text{—} \circledast$$

In steady state, $u_t = 0 \Rightarrow u_{xx} = 0$

Integrate w. r. to 'x'

$$u_x = a$$

Integrate w. r. to 'x'

$$u = ax + b.$$

How many conditions are required to solve.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}. \text{ Three.}$$

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7) In a diffusion equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ what does α^2 stand for.

$\alpha^2 = \frac{k}{s\rho}$ called diffusivity of the material of the bar.

$k \to$ Thermal conductivity of the material

$s \to$ specific heat capacity of the material.

$\rho \to$ density of the material.

8) Write all variable seperable soln of heat equation $u_t = \alpha^2 u_{xx}$.

$y(x,t) = (A \cos px + B \sin px) e^{-\alpha^2 p^2 t}$

$y(x,t) = (A_1 x + A_2) \quad A_3$

$y(x,t) = (A_4 e^{px} + A_5 e^{-px}) \quad (A_6 e^{-\alpha^2 p^2 t})$

9) state any two laws which are assumed to derive one D ~~eqn~~ heat equation.

(i) Heat flows from higher to lower temperature

(ii) The rate at which heat flows across any area is propotional to the area and to the temperature gradient normal to the curve This constant of propotionally is known as the thermal conductivity (k) of the material. It is known as Fourier laws

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of heat equation.

Q7) What is the basic difference between the solution of 1D wave equation and 1D heat equation.

The suitable solution of 1D wave equation is peridic in nature. But the suitable solution of 1D heat equation is not periodic in nature.

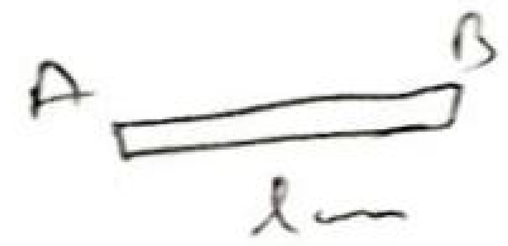
The suitable solution of 1D heat equation contains exponential term. But the suitable solution of 1 Dimentional wave equation does not contain exponential term.

Q) State 1D heat equation with initial and boundary condition.

$u(0,t) = 0$, ~~$0 \le x \le l$~~ $t \geq 0$

$u(l,t) = 0$ ~~$0 \le x \le l$~~ $t \geq 0$

$u(x,0) = 0$ $0 \le x \le l$.

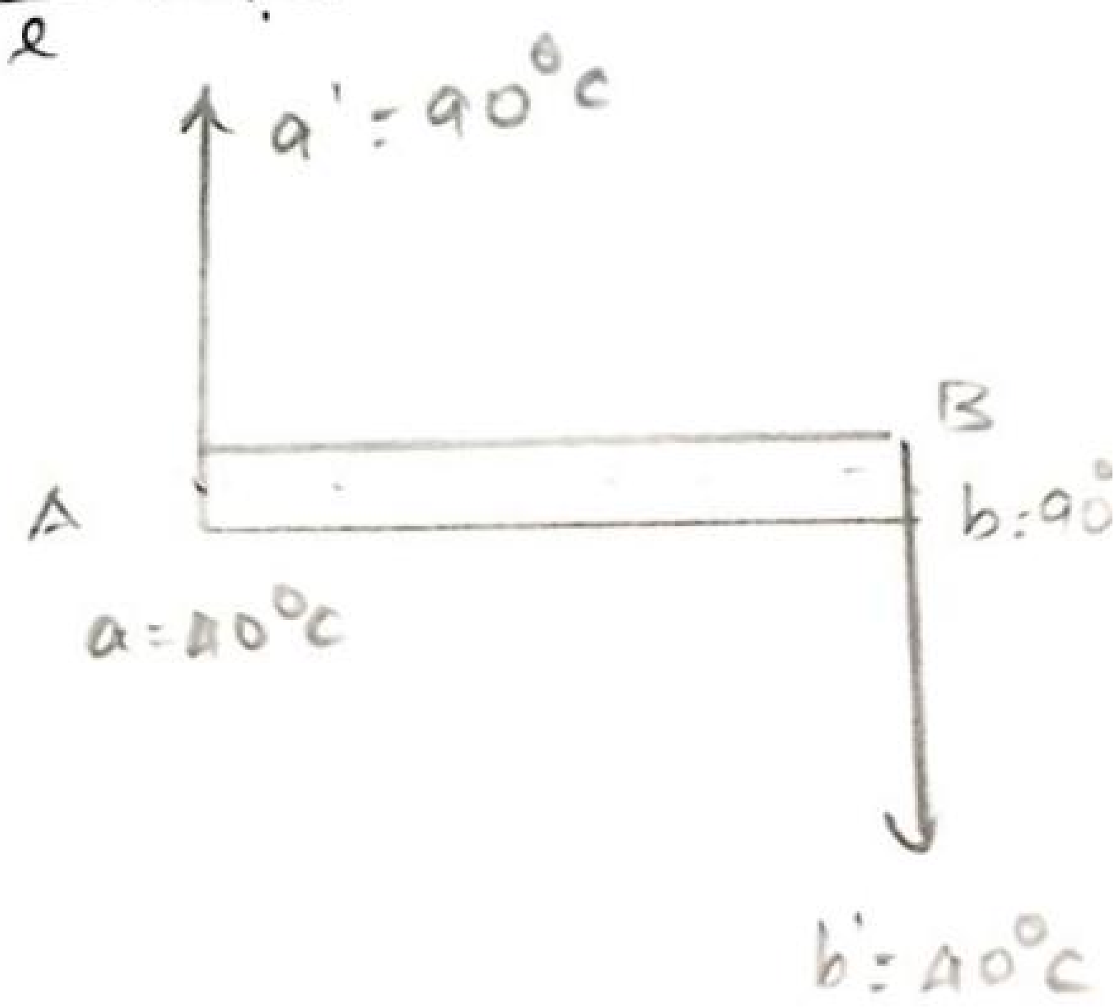


Steady state conditions with non-zero boundary conditions.

The ends A and B of rod l cm long have the temperature $40°C$ and $90°C$ until this steady state conditions prevail. The temperature at A is suddenly rasied to $90°C$ and at the same time the temperature at B is lowered to $40°C$. Find the temperature distribution in the rod at time t. Also show that the temperature at mid point of the rod remains unaltered for all time regardless of the material of the rod.

1 D heat eqn is $u_t = \alpha^2 u_{xx}$

Steady state temperature $u(x) = \frac{b-a}{l} x + a$



length = l cm

$b = 90°c$

$a = 40°c$

$u(x) = \frac{50}{l} x + 40$

After steady state the B.C's in the unsteady state are.

a) $u(0,t) = 90$

b) $u(l,t) = 40$

c) $u(x,0) = \frac{50x}{l} + 40, \quad 0 \le x \le l.$

The required temperature distribution

$u(x,t) = u_s(x) + u_t(x,t)$ ——— (*)

Where $u_s(x)$ is the steady state solution & $u_t(x,t)$ is the transient state solution.

To find $u_s(x)$

$$u_s(x) = \frac{b'-a'}{l} x + a'$$

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$b' = 40°c$

$a' = 90°c$

length = l cm

$$u_s(x) = \frac{-50x}{l} + 90$$

To Find $u_t(x,t)$

From (*), $u_t(x,t) = u(x,t) - u_s(x)$

Boundary conditions

(i) $u_t(0,t) = u(0,t) - u_s(0)$

$= 90 - 90$

$= 0$

(ii) $u_t(l,t) = u(l,t) - u_s(l)$

$= 40 - 40$

$= 0$

(iii) $u_t(x,0) = u(x,0) - u_s(x)$

$$= \frac{50x}{l} + 40 - \left(\frac{-50x}{l} + 90\right)$$

$$u_t(x,0) = \frac{100x}{l} - 50$$

The suitable solution is

$$u_t(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2p^2t} \quad \text{—— (1)}$$

Apply (i) in (1)

$A\, e^{-\alpha^2p^2t} = 0 \qquad \therefore e^{-\alpha^2p^2t} \neq 0$

$\boxed{A = 0}$

Sub in (1)

$$u_t(x,t) = B \sin px \; e^{-\alpha^2 p^2 t} \quad \text{—— (2)}$$

Apply (ii) in (2)

$$B \sin pl \; e^{-\alpha^2 p^2 t} = 0$$

$$\sin pl = 0 \qquad \therefore B \neq 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

sub this in (1)

$$u_t(x,t) = B \sin \frac{n\pi x}{l} \; e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

Generalising the above equation, we got

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \; e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{—— (3)}$$

Apply (iii) in (3)

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100x}{l} - 50$$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{100}{l} \int_0^l \left(\frac{2x}{l} - 1\right) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{100}{l} \left[\left(\frac{2x}{l} - 1\right) \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) - \frac{2}{l} \left(\frac{-\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2} \right) \right]_0^l$$

$$= \frac{100}{l} \left[\frac{-l}{n\pi} (-1)^n - \frac{l}{n\pi} \right] = \frac{-100}{\cancel{l}} \frac{\cancel{l}}{n\pi} \left[(-1)^n + 1 \right]$$

$= \frac{-100}{n\pi} \cdot 2$ if n is even

$B_n = \frac{-200}{n\pi}$ if n is even

Sub this value in (3)

$$u_t(x,t) = \sum_{n=2,4\ldots}^{\infty} \left(\frac{-200}{n\pi}\right) \sin \frac{n\pi x}{l} \, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u_t(x,t) = -\frac{\cancel{200}\ 100}{\pi} \sum_{n=1}^{\infty} \frac{1}{\cancel{2}n} \sin \frac{2n\pi x}{l} \, e^{-\frac{4\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{-100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l} \, e^{-\frac{4\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = -\frac{50x}{l} + 90 - \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l} \, e^{-\frac{4\alpha^2 n^2 \pi^2 t}{l^2}}$$

~~If~~ the temperature remains unaltered.

put, $x = l/2$

$\boxed{u(l/2, t) = 65}$. (Ans)

PROBLEMS BASED ON

FINITE PLATE

The boundary value problems governing the steady state temperature distribution in a flat thin, square plate is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\quad 0 < x < a,\ 0 < y < a$

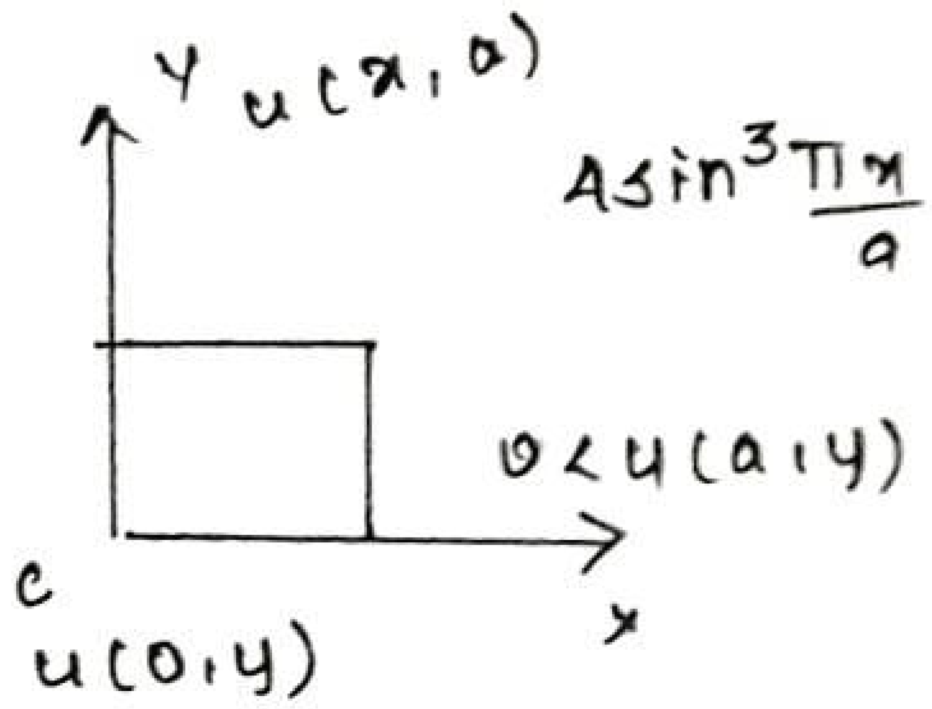
$u(x,0) = 0$, $\quad u(x,a) = 4\sin^3\left(\frac{\pi x}{a}\right)$; $\quad 0 < x < a$.

$u(0,y) = 0$, $\quad u(a,y) = 0$ $\quad 0 < y < a$

Find the steady state temperature distribution in the plate.

The steady state 2D heat eqn is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- } (*)$$



Given boundary equation.

(i) $u(0,y) = 0$ $\quad 0 < y < a$.

(ii) $u(a,y) = 0$ $\quad 0 < y < a$

(iii) $u(x,0) = 0$ $\quad 0 < x < a$

(iv) $u(x,a) = \frac{4\sin^3 \pi x}{a}$ $\quad 0 < x < a$.

The suitable soln of equation (*) is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad \text{--- } (1).$$

Apply (i) in (1)

$A(Ce^{py} + De^{-py}) = 0$ $\qquad \therefore Ce^{py} + De^{-py} \neq 0$

$\boxed{A = 0}$

Sub the value in ①.

$$u(x,y) = B \sin px [C e^{py} + D e^{-py}] \quad \text{②}$$

$$\sin pa = 0$$

$$pa = n\pi$$

$$p = \frac{n\pi}{a}$$

Subst this value in ②

$$u(x,y) = B \sin \frac{n\pi x}{a} \left[C e^{\frac{n\pi y}{a}} + D e^{-\frac{n\pi y}{a}} \right] \quad \text{——— ③}$$

Apply (iii) in ③

$$B \sin \frac{n\pi x}{a} (C + D) = 0$$

$$C + D = 0$$

$$\boxed{D = -C}$$

Subst this value is ③

$$u(x,y) = B \sin \frac{n\pi x}{a} \left[C e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right]$$

$$= BC \sin \frac{n\pi x}{a} \left[e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right]$$

$$= BC \sin \frac{n\pi}{a} x \cdot 2 \cdot \sinh \frac{n\pi y}{a}.$$

Generalising the above equ, we get.

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi y}{a} \quad \text{——— ④}$$

Apply (iv) in ④.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh n\pi = 4 \sin^3 \frac{\pi x}{a}$$

$$= 3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a}.$$

Equating like terms.

$$B \sinh \pi = 3$$

$$B_1 = \frac{3}{\sinh \pi}$$

$$\boxed{B_2 = 0}$$

$$B_3 \sinh 3\pi = -1$$

$$B_3 = \frac{-1}{\sinh 3\pi}$$

$$\boxed{B_3 = -\operatorname{cosech} 3\pi}$$

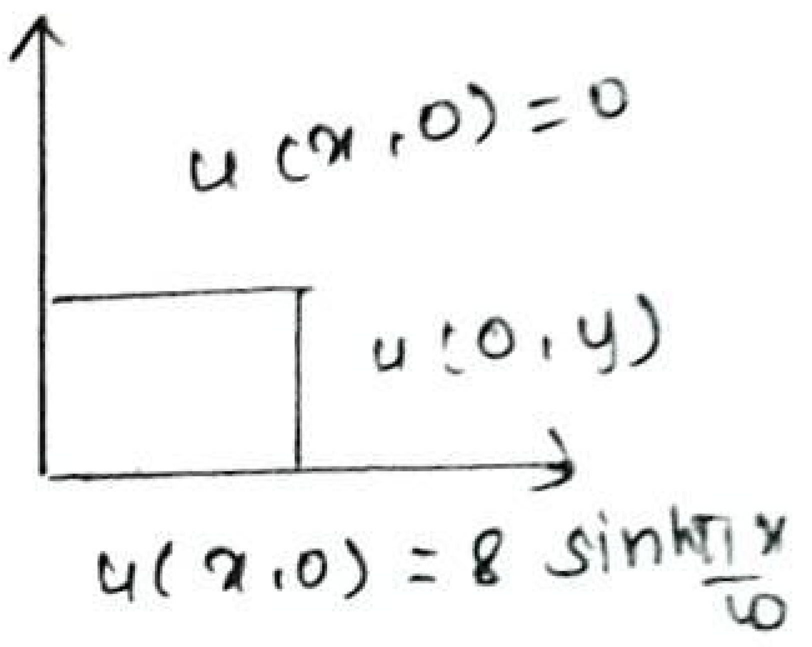
Subst the value of B_1 and B_3 in eq (4).

$$u(x,y) = 3 \operatorname{cosech} \pi \sin \frac{\pi x}{a} \sinh \frac{\pi y}{a} - \operatorname{cosech} 3\pi \sin \frac{3\pi x}{a} \sinh \frac{3\pi y}{a}.$$

PROBLEMS BASED ON INFINITE PLATE

A rectangular plate with insulated surface is 10 cm wide. so long compared to its width that may be considered length. If the temperature along short edge $y=0$ is given $u(x,0) = 8 \sin \frac{n\pi x}{10}$ where $0 < x < 10$ while the two long edge $x=0$ $ $x=10$ as well as the others a short edge kept at $0^\circ C$. Find the steady state temperature functions (x,y)

The 2D steady state heat eqn is



$u_{xx} + u_{yy} = 0$ —— (*) .

Given boundary conditions

(i) $u(0,y) = 0 \quad \forall y$

(ii) $u(10,y) = 0 \quad \forall y$

(iii) $u(x,\infty) = 0 \quad 0 < x < 10$

(iv) $u(x,0) = 8 \sin \frac{\pi x}{10} \quad 0 \le x \le 10$

The suitable soln of (*) is

$$u(x,y) = (A\cos px + B\sin px)\ (Ce^{py} + De^{-py})$$

Apply (ii) in (2).

$$B\sin 10p \left[Ce^{py} + De^{-py}\right] = 0$$

$$\sin 10p = 0$$

$$\boxed{p = n\pi/10}$$

Subst the value of p in (2)

$$u(x,y) = B\sin\frac{n\pi x}{10}\left[Ce^{\frac{n\pi y}{10}} + De^{-n\pi y/10}\right] \quad \text{—(3)}.$$

Apply (iii) in (3)

$$B\sin\frac{n\pi x}{10}\left[Ce^{\infty} + De^{-\infty}\right] = 0$$

$$B\sin\frac{n\pi x}{10}\ Ce^{\infty} = 0$$

$\left[\therefore e^{\infty} \neq 0 \ ;\ B \neq 0 \quad \therefore \sin\frac{n\pi x}{10} \neq 0\right]$

$$\boxed{C = 0}$$

$$u(x,y) = B\sin\frac{n\pi x}{10}\ De^{-\frac{n\pi y}{10}} \quad \text{—— (4)}.$$

Apply (iv) in (4)

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} = 8 \sin \frac{n\pi x}{10}$$

Equating like terms

$B_1 = 8, \quad B_n = 0 \quad$ for $n = 2, 3 \ldots$

Subst the value of B_n in (A)

$$\boxed{u(x,y) = 8 \sin \frac{\pi x}{10} e^{-\pi y/10}}$$

A rectangular plate with insulated surface is 10 cm wide and 80 cm long compared to its width that it may be considered infinite length without introducing appreciable error, the temperature at short edge, $y=0$ is given by

$u = 20x$ for $0 \le x \le 5$
$20(10-x)$ for $5 \le x \le 10$ } and all the other three edges are kept at $0°c$ Find the steady state temperature at any point in the plate

The steady state 2D heat eqn is

$$u_{xx} + u_{yy} = 0$$

Boundary conditions

(i) $u(0,y) = 0 \quad \forall y$

(ii) $u(x,y) = 0 \quad \forall y$

(iii) $u(x,\infty) = 0 \quad 0 < x < 10$

(iv) $u(x,0) = \begin{cases} 20x & 0 \le x \le 5 \\ 20(10-x) & 5 \le x \le 10 \end{cases}$

The suitable soln of (A) is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad \text{—— (B)}$$

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Apply (i) in ①

$A[ce^{py}+De^{-py}]=0$ $\therefore ce^{py}+De^{-py}\neq 0$

$\boxed{A=0}$

Sub the values in ①.

$u(x,y) = B\sin px\,[ce^{py}+De^{-py}]$ — ②

Apply (ii) in ②

$B\sin 10p\,(ce^{py}+De^{-py})=0$

$\sin 10p=0$

$10p=n\pi$

$p=\frac{n\pi}{10}$

Sub this value in ②

$u(x,y)=B\sin\frac{n\pi x}{10}\left[ce^{\frac{n\pi y}{10}}+De^{-n\pi y/10}\right]$

Apply (iii) in ③

$B\sin\frac{n\pi x}{10}\,(ce^{\infty}+De^{-\infty})=0$

$B\sin\frac{n\pi x}{10}\,ce^{\infty}=0$ $\therefore e^{\infty}\neq 0$

$\boxed{c=0}$ $[e^{\infty}\neq 0,\ B\neq 0]$

Subst this value in ③

$u(x,y)=B\sin\frac{n\pi x}{10}\,De^{-\frac{n\pi y}{10}}\sin\frac{n\pi x}{10}=0$

Generalising the above equation we get

$u(x,y)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{10}\,e^{-\frac{n\pi y}{10}}$ — Ⓐ.

Apply (iv) in ④

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$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} = \begin{cases} 20x & 0 < x < 5 \\ 20(10-x) & 5 < x < 10 \end{cases}$$

Where,

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{10} \left\{ \int_0^5 20x \sin \frac{n\pi x}{10}\, dx + \int_5^{10} 20(10-x) \sin \frac{n\pi x}{10}\, dx \right.$$

$$= \frac{40}{10} \left[\int_0^5 x \sin \frac{n\pi x}{10}\, dx + \int_5^{10} (10-x) \sin \frac{n\pi x}{10}\, dx \right]$$

$$= 4 \left[x \left(\frac{-\cos \frac{n\pi x}{10}}{n\pi/10} \right) - \left(\frac{-\sin \frac{n\pi x}{10}}{n^2\pi^2/100} \right) \right]_0^5 + \left[(10-x) \left(\frac{-\cos \frac{n\pi x}{10}}{n\pi/10} \right) \right.$$

$$\left. - (-1) \left(\frac{-\sin n\pi x/10}{n^2\pi^2/100} \right) \right]_5^{10}$$

$$= 4 \left[\cancel{\frac{-50}{n\pi} \cos n\pi/2} + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{50}{n\pi} \cos n\pi/2} \right.$$

$$\left. + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$B_n = 4 \left[\frac{200}{n^2\pi^2} \sin n\pi/2 \right] \text{ if } n \text{ is odd.}$$

$$B_n = \frac{800}{n^2\pi^2} \sin n\pi/2 \text{ if } n \text{ is odd.}$$

Subst the value of B_n in (A)

$$u(x,y) = \frac{800}{\pi^2} \sum_{n=1,3,\dots} 1/n^2 \sin n\pi/2 \; \sin \frac{n\pi x}{10} \cdot e^{-n\pi y/10}$$

$$u(x,y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} 1/(2n-1)^2 \; \sin \frac{(2n-1)\pi}{2} \; \sin \frac{(2n-1)\pi x}{10} \; e^{-\frac{(2n-1)\pi x}{10}}$$

SECOND ORDER LINEAR PARTIAL DIFFERNTIAL EQUATIONS

General Form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = F(x,y)$$

(or)

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x - E u_y + Fu = F(x,y) \quad \text{--- (1)}$$

Eqn (1) is elliptic if $B^2 - 4AC < 0$

Equation (1) is parabolic if $B^2 - 4AC = 0$

Equation (1) is hyperbolic if $B^2 - 4AC > 0$

1) classified the laplace equation (or) given example of elliptic type equation.

$$u_{xx} + u_{yy} = 0$$

here $A = 1$, $B = 0$, $C = 1$

$$B^2 - 4AC = -4 < 0$$

$\therefore$ Laplace equation is elliptic

classify the poisson equation.

$$u_{xx} + u_{yy} = f(x,y)$$

Here $A = 1$ $B = 0$ $C = 1$

$$B^2 - 4AC = -4 < 0$$

$\therefore$ poission equation is ~~parabolic~~. elliptic classify the one Dimentional heat equation

$$u_t = \alpha^2 u_{xx}.$$

$$\alpha^2 u_{xx} - u_t = 0$$

Here $A = \alpha^2$ $B = 0$ $C = 0$

$$B^2 - 4AC = 0$$

$\therefore$ 1 D heat equation (or) heat condonsion is parabolic.

Classify one Dimentional wave equation.

$$u_{tt} = \alpha^2 u_{xx}$$

$$\alpha^2 u_{xx} - u_{tt} = 0$$

Here $A = \alpha^2$ $B = 0$ $C = -1$

$$B^2 - 4AC = 0$$

$$4\alpha^2 > 0$$

$\therefore$ 1 D wave equation is hyperbolic.

UNIT IV

Dr. K Iyappan

Fourier Transform

Write Fourier Transform pair

(i) Fourier transform (complex Fourier transform or infinite Fourier transform)

The Fourier transform of $f(x)$ is defined as

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx = F(s)$$

(ii) Inverse Fourier transform

The Inverse Fourier transform of Fourier transform function $F(s)$ is defined as

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \, e^{-isx} \, ds$$

(iii) Parseval's Identity (Energy Theorem)

If $F(s)$ is the Fourier transform of $f(x)$ then $\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$

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1) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0 & |x| > a > 0 \end{cases}$

Hence find the value of integrals

$$\int_0^\infty \frac{\sin x}{x}\, dx \quad \& \quad \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$$

Given

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx. \qquad e^{isx} = \cos sx + i \sin sx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx. \qquad \left[\because \int_{-a}^{a} \sin sx\, dx = 0 \text{ because sin is odd}\right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx\, dx$$

$$= \sqrt{2/\pi} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{2/\pi} \left[\frac{\sin as}{s}\right] = F(s)$$

Taking inverse fourier Transform

$$F^{-1}[F(s)] = f(x)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = f(x)$$

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$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi}\, \frac{\sin as}{s} e^{-isx} ds = 1$$

put $x = 0$

$$1/\pi \int_{-\infty}^{\infty} \frac{\sin as}{s} ds = 1$$

$$2/\pi \int_{0}^{\infty} \frac{\sin as}{s} ds = 1$$

put $a = 1$

$$\int_{0}^{\infty} \frac{\sin s}{s} ds = \pi/2$$

put $s = x$.

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \pi/2$$

using parseval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

$$\Rightarrow 2/\pi \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2} ds = \int_{-a}^{a} dx$$

$$\Rightarrow 2/\pi\, \cancel{2} \int_{0}^{\infty} \frac{\sin^2 as}{s^2} ds = \cancel{2} \int_{0}^{a} dx. = [x]_0^a$$

$$\Rightarrow 2/\pi \int_{0}^{\infty} \frac{\sin^2 as}{s^2} ds = a$$

Put a=1

$$\frac{2}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = 1$$

$$\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = \pi/2$$

put $s = x$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi/2$$

Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$

hence evaluate the following integrals (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$

(ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

Given

$$f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1-|x|\, e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx$$

$$= \sqrt{2/\pi} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{2/\pi} \left[\frac{-\cos s}{s^2} + 1/s^2\right]$$

$$= 2\sqrt{2/\pi} \left[\frac{\sin^2 s/2}{s^2}\right]$$

$$= F(s)$$

Taking inverse fourier ransform

$$F^{-1}[F(s)] = f(x)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds = f(x)$$

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$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{2/\pi} \; \frac{\sin^2 s/2}{s^2} \, e^{isx}$$ $dx = 1 - |x|$

put $x = 0$

$$= 2/\pi \int_{-\infty}^{\infty} \frac{\sin^2 s/2}{s^2} \, ds = 1$$

$$4/\pi \int_{0}^{\infty} \frac{\sin^2 s/2}{s^2} \, ds = 1$$

$s = 2t$

$ds = 2\,dt$

put $s/2 = t$

$$\frac{4}{\pi} \int_{0}^{\infty} \frac{\sin^2 t}{4t^2} \, 2\,dt$$

$$\int_{0}^{\infty} \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2}$$

using parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \frac{\sin^4 s/2}{s^4} \, ds = \int_{-1}^{1} (1 - |x|)^2 \, ds$$

$$= 2 \int_{0}^{1} (1 - x^2)^2 \, dx$$

$$= 2 \int_{0}^{1} (1 - 2x + x^2) \, dx.$$

$$= 2 \left[x - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^1$$

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$$= 8\left[1 - 1 + \tfrac{1}{3}\right]$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4 s/2}{s^4}\, ds = 8/3$$

$$\frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 s/2}{s^4}\, ds = 8/3$$

$$\int_{0}^{\infty} \frac{\sin^4 s/2}{s^4}\, ds = \frac{8\pi}{16 \times 3}$$

put $s/2 = t$

$s = 2t$

$ds = 2\, dt$

$$\int_{0}^{\infty} \frac{(\sin t)^4}{\cancel{16} t^4}\, \cancel{2}\, dt = \frac{\cancel{8}\pi}{16 \times 3}$$

$$\int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \pi/3$$

tion given by

Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2 & , \quad |x| < a \\ 0 & , \quad |x| > a > 0 \end{cases}$$ is $2\sqrt{2/\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)$

Hence deduced that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \pi/4$ also

Find the value $\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6} dt$

$$f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a > 0 \end{cases}$$

$$[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \cos sx\, dx. = \frac{1}{\sqrt{2\pi}} 2 \int_0^a (a^2 - x^2) \cos sx\, dx$$

$$= \sqrt{2/\pi} \left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) \right]_0^a.$$

$$= \sqrt{2/\pi} \left[-2a \frac{\cos as}{s^2} + 2 \frac{\sin as}{s^3} \right]$$

$$= 2\sqrt{2/\pi} \left[\frac{\sin as}{s^3} - \frac{a \cos as}{s^2} \right]$$

$$= 2\sqrt{2/\pi}\left[\frac{\sin as - as\cos as}{s^3}\right] = F(s)$$

Taking inverse transform

$$F^{-1}[F(s)] = f(x)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left(\frac{+\sin as - as\cos as}{s^3}\right) e^{-isx}\, ds = a^2 - x^2$$

Put $x = 0$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right) ds = a^2$$

put $a = 1$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = 1$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = 1$$

Put $s = t$

$ds = dt$

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right) dt = \pi/4$$

using parseval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{(\sin as - as\cos as)^2}{s^6}\,ds = \int_{-a}^{a}(a^2-x^2)^2$$

$$\frac{8}{\pi}\,2\int_{0}^{\infty}\frac{(\sin as - as\cos as)^2}{s^6}\,ds = 2\int_{0}^{a}(a^4-2a^2x^2+x^4)\,dx$$

$$= \left[a^4x - 2a^2x^3/3 + x^5/5\right]_0^a$$

$$= \frac{15a^5 - 10a^5 + 3a^5}{15}$$

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{(\sin as - as\cos as)^2}{s^6}\,ds = \frac{8a^5}{15}$$

Put $a = 1$

$$\int_{0}^{\infty}\frac{(\sin s - s\cos s)^2}{s^6}\,ds = \pi/15$$

put $s = t$

$ds = dt$

$$\int_{0}^{\infty}\frac{(\sin t - t\cos t)^2}{t^6}\,dt = \pi/15$$

=

$*$ $\displaystyle\int_{0}^{\infty}\frac{\sin x - x\cos x}{x^3}\cos x/2\,dx$ prove

Taking Inverse fourier transform

$$F^{-1}[F(s)] = f(x)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{2/\pi}\left[\frac{\sin as - as\cos as}{s^3}\right]e^{-isx}\,ds = a^2-x^2$$

$$4/\pi\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\,ds = a^2-x^2$$

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$$\frac{A}{\pi}\int_0^\infty \frac{\sin as - as\cos as}{s^3}\cos sx\,ds = a^2 - x^2$$

put $a=1$

$$\frac{A}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos x\,ds = 1 - x^2$$

put $x = 1/2$

$$A/\pi\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos s/2\,ds = 3/4$$

put $s = x$.

$$\int_0^\infty \frac{\sin x - x\cos x}{x^3}\cos x/2\,dx = 3\pi/16.$$

Find the fourier transform of $e^{-a|x|}$ if $a>0$ deduced that $\int_0^\infty 1/(x^2+a^2)^2\,dx = \pi/4a^3$ and $\int_0^\infty 1/(x^2+a^2)\,dx = \pi/2a$.

$$f(x) = e^{-a|x|} \quad a>0$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{isx}\,dx. = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx.$$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$F\left(f(x)\right) = \sqrt{2/\pi}\ \frac{a}{a^2+s^2} \quad F(s)$$

Taking inverse fourier transform

$$F^{-1}\left[F(s)\right] = f(x)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi}\ \frac{a}{a^2+s^2}\ e^{-isx}\ dx = e^{-a|x|}$$

put $x = 0$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi}\ \frac{a}{a^2+s^2}\ ds = 1$$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{a}{a^2+s^2}\ ds = 1$$

$$\int_{0}^{\infty} \frac{1}{a^2+s^2}\ ds = \pi/2a.$$

Put $s = x$.

$$\int_{0}^{\infty} \frac{1}{a^2+x^2}\ dx = \pi/2a.$$

using parseval's Idenity

$$\int_{-\infty}^{\infty} |F(s)|^2\ ds = \int_{-\infty}^{\infty} |F(x)|^2\ dx.$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{a^2}{(a^2+s^2)^2}\ ds = \int_{-\infty}^{\infty} \left[e^{-a|x|}\right]^2 dx.$$

$$2/\pi\ \ 2\int_{0}^{\infty} \frac{a^2}{(a^2+s^2)^2}\ ds = 2\int_{0}^{\infty} \left[e^{-ax}\right]^2 dx.$$

$$= \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{a^2}{(a^2+s^2)^2} \, ds = 1/8a.$$

$$\int_0^{\infty} \frac{1}{(a^2+s^2)^2} \, ds = \frac{\pi}{4a^3}$$

put $s = x$.

$$\int_0^{\infty} \frac{1}{(a^2+x^2)^2} \, dx = \pi/4a^3$$

x^s

Verify Parseval's theorem $f(x) = \begin{cases} 0 & ; x < 0 \\ e^{-x} & ; x > 0 \end{cases}$

Given

$$f(x) = \begin{cases} 0 & ; x < 0 \\ e^{-x} & ; x > 0 \end{cases}$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-is)x} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-is)x}}{-(1-is)} \right]_0^{\infty}$$

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$$F(S) = \frac{1}{\sqrt{2\pi}} \frac{1}{1-is}$$

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

$$\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \frac{1}{(1-is)}\right] \left[\frac{1}{\sqrt{2\pi}} \frac{1}{1+is}\right] ds = \int_{0}^{\infty} (e^{-x})^2 \, dx.$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+s^2} \, ds = \int_{0}^{\infty} e^{-2x} \, dx.$$

$$\cancel{2} \cdot \frac{1}{\cancel{2}\pi} \int_{0}^{\infty} \frac{1}{1+s^2} \, ds = \int_{0}^{\infty} e^{-2x} \, dx.$$

$$\frac{1}{\pi} \left[\tan^{-1} s\right]_0^{\infty} = \left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$\frac{1}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{2} = 1/2$$

$$1/2 = 1/2$$

Hence parseval's theorem is verified.

Prove that $e^{-x^2/2}$ is self reciprocal

Soln.

$$f(x) = e^{-x^2/2}$$

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx.$$

$$F\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2 - isx)} \, dx.$$

$a^2 = x^2/2$

$a = x/\sqrt{2}$

$2ab = isx$

$\sqrt{2} \cdot 2 \frac{x}{\sqrt{2}} b = isx$

$b = \frac{is}{\sqrt{2}}$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\sqrt{2} - is/\sqrt{2})^2} \, dx.$$

Put $\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}} = t$

$$\frac{dx}{\sqrt{2}} = dt$$

$$dx = \sqrt{2}\, dt$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \sqrt{\pi}$$

$$F\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

Hence $e^{-x^2/2}$ is self reciprocal.

Find Fourier transform of $e^{-a^2x^2}$ hence prove that $e^{-x^2/2}$ is self reciprocal.

$$f(x) = e^{-a^2x^2}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx.$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax - is/2a)^2}\, dx.$$

put $ax - \frac{is}{2a} = t$

$a\, dx = dt$

$dx = dt/a.$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$(e^{-x^2}] = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$ put $a = 1/\sqrt{2}$

$$F\left[e^{-x^2/2}\right] = e^{-s^2/2}.$$

Find Fourier transform of $e^{-a|x|}$ $a>0$ hence deduce that

$$F\left[x e^{-a|x|}\right] = i\sqrt{2/\pi}\ \frac{2as}{(a^2+s^2)^2}.$$

Soln.

$$f(x) = e^{-a|x|}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx.$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} \cos sx\, dx.$$

$$= \sqrt{2/\pi} \left[\frac{e^{-ax}}{a^2+s^2}\left[(-a\cos sx + s\sin sx)\right]\right]_0^{\infty}$$

$$= \sqrt{2/\pi} \left[\frac{a}{a^2+s^2}\right] = F(s)$$

$$F\left[x\, f(x)\right] = -i \frac{d}{ds} F\left[f(x)\right]$$

$$F\left[x e^{-a|x|}\right] = -i \frac{d}{ds} F\left[e^{-a|x|}\right]$$

$$= -i\sqrt{2/\pi}\ d/ds \left(\frac{a}{a^2+s^2}\right)$$

$$= -ai\sqrt{2/\pi}\ d/ds\ (a^2+s^2)^{-1}$$

$$= -ai\sqrt{2/\pi}\ (-1)\ (a^2+s^2)^{-2}\, 2s$$

$F[xf(x)] = i\sqrt{2/\pi}\left(\frac{2as}{(a^2+s^2)^2}\right)$

PROPERTIES

.Find the fourier transform of $f(x-a)$

(or)

If the fourier transform $f(x)$ is $F(s)$, show that F.T of $f(x-a)$ is $e^{ias}F(s)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{isx}\,dx.$$

put $x-a=t$

$x = a+t$

$dx = dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\,dt$$

$$= e^{+isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{ist}\,dt$$

$$F[f(x-a)] = e^{isa}F(s)$$

~~change of scale property~~.

(ii) Find the fourier transform of $[x\, f(x)]$ [first derivative of fourier transform]

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx.$$

Diff w.r.t 's' once

$$\frac{d\,[F(s)]}{ds} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)\, dx$$

$$= i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, f(x)\, e^{isx}\, dx.$$

$$= i\, F\,[x\, f(x)]$$

$$F\,[x\, f(x)] = 1/i \; d/ds\,[F(s)]$$

$$\boxed{F\,[x\, f(x)] = -i\; d/ds\,[F(s)]}$$

Find the fourier transform of $x^n f(x)$ [n^{th} derivati[ve]
of fourier transform]

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

Diff w.r.t 's' n times

$$\frac{d^n\,[F(s)]}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)^n\, dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x)\, e^{isx}\, dx.$$

$$= (i)^n\, F\,[x^n f(x)]$$

$$F\,[x^n f(x)] = 1/i^n \; d^n/ds^n\,[F(s)]$$

$$F\,[x^n f(x)] = (-i)^n\, d^n/ds^n\, F(s)$$

(iv) Properties:

Modulation theorom:

State and prove Modulation theorem (or) find the Fourier transform of $f(x) \cos ax$.

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{isx} \, dx.$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{+i(s+a)x} + e^{i(s-a)x}\right] dx.$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x} \, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x} \, dx\right]$$

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

(v) property

Shifting property ශ ළ.

Find the fourier transform of $f(x)\, e^{iax}$.

$$F[f(x)\, e^{iax}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iax}\, e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x} \, dx.$$

$$F[f(x)\, e^{iax}] = F(s+a)$$

(Vi) property: Change of scale property

Find the fourier transform of $f(ax)$

S.T $F[f(ax)] = \frac{1}{|a|} F(s/a)$ (or)

Soln

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx.$$

put $ax = t$

$x = t/a.$

$dx = dt/a.$

case (i)

(a is positive)

when $x = -\infty$, $t = -\infty$

$x = \infty$. $t = \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\,t/a}\, dt/a.$$

$$F[f(ax)] = 1/a \; F(s/a) \quad \text{--- (1)}$$

case (ii)

a is negative

when $x = -\infty$, $t = \infty$

$x = \infty$, $t = -\infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{is\,t/a}\, dt/a.$$

$$= -1/a \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a}\, dt$$

$$F[f(ax)] = -1/a \; F(s/a) \quad \text{--- (2)}.$$

From (1) & (2).

$$F[f(ax)] = \frac{1}{|a|} F(s/a)$$

Definition

Convolution.

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

State and prove convolution theorem.

The Fourier transform of convolution of two functions $f(x)$ and $g(x)$ is equal to the product of their Fourier transform

i.e $F[f(x) * g(x)] = F(s)\, G(s)$

Proof

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x) * g(x)]\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx}\; dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

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$$= G(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$F[f(x) * g(x)] = G(s) \quad F(s)$$

State and prove parseval's Identity

If $f(s)$ is the Fourier transform of $f(x)$

then $\int_{-\infty}^{\infty} |F(x)^2|\, dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$ conjugate symmetr

Property.

$$F\left[\overline{f(-x)}\right] = \overline{F(s)}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx.$$

put $x = -t \quad dx = -dt$

where $x = -\infty \, , \; t = \infty$

$x = \infty \, , \; t = -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, (-dt)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$= F\left[\overline{f(-t)}\right]$$

$$\overline{F(s)} = F\left[\overline{f(-x)}\right]$$

Proof of main theorem

By convolution theorem

$F[f(x) * g(x)] = F(s)\, G(s)$

Taking IFT,

$f(x) * g(x) = F^{-1}[F(s)\, G(s)]$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx}\, ds$$

put $x = 0$

$$\int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \int_{-\infty}^{\infty} F(s)\, G(s)\, ds \quad \text{—— (1)}$$

let $g(-t) = \overline{f(t)}$

$g(t) = \overline{f(-t)}$

$F[g(t)] = F[\overline{f(-t)}]$

$G(s) = \overline{F(s)}$ By conjugate symmetric property

Sub the values of $g(-t)$ & $G(s)$ in (1)

$$\int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

changing the dummy variable

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Sine & cosine:

Write fourier cosine transform pair.

The Fourier cosine transform of $f(x)$ is defined as

$$F_c\,[f(x)] = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos sx \, dx = F_c(s)$$

Inverse Fourier Cosine transform.

The Inverse fourier cosine transform of $F_c(s)$ is defined as

$$F_c^{-1}\,[F_c(s)] = f(x) = \sqrt{2/\pi} \int_0^{\infty} F_c(s) \cos sx \, ds$$

Fourier Sine transform

The Fourier Sine transform of $f(x)$ is defined as

$$F_s[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx = F_s(s)$$

Inverse Fourier sine transform

The Inverse Fourier sine transform of $F(s)$ is defined as

$$F_s^{-1}[F_s(s)] = f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

Parseval's Identity

Parseval's Identity in a singale function in a cosine transform

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$$

Parseval's Identity for a singale function in sine transform

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$$

Parseval's Identity for two functions in cosine transform:

If $F_c(s)$ and $G_c(s)$ are fourier cosine transforms of $f(x)$ & $g(x)$ respectivly

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$$

Parseval's Identity for two functions in sine transforms

If $F_s(s)$ & $G_s(s)$ are Fourier sine transforms of $f(x)$ & $g(x)$ repectivly

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s(s)\, G_s(s)\, ds$$

Find Fourier sine and cosine transform of e^{-ax} and hence deduced the inversion formula also evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$ and $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx$

Given

$$f(x) = e^{-ax} \quad a > 0$$

$$F_s[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[e^{-ax}] = \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{2/\pi} \left[\frac{e^{-ax}}{s^2+a^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$F_s[e^{-ax}] = \sqrt{2/\pi} \; s/s^2+a^2$$

Taking inverse FST $F_s^{-1}[F_s(s)] = f(x)$

$$\sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, ds = f(x)$$

$$\sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \frac{s}{s^2+a^2} \sin sx \, ds = e^{-ax}$$

$$\frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} ds = e^{-ax}$$

put $s = x$ & $x = \alpha$.

$$\int_0^\infty \frac{x \sin \alpha x}{x^2+a^2} dx = \pi/2 \; e^{-\alpha a}$$

using parseval's Identity for sine transform.

$$\int_0^\infty |F_s(s)|^2 ds = \int_0^\infty |f(x)|^2 dx.$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds = \int_0^\infty e^{-2ax}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \pi/2 \cdot 1/2a$$

Put $s = x$.

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \pi/4a.$$

Cosine transform.

$$F_c[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx.$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin x)\right]_0^\infty$$

$$F_c[f(x)] = \sqrt{2/\pi}\ a/a^2+s^2 = f(s)$$

Taking inverse FCT

$$\sqrt{2/\pi}\int_0^\infty F_c(s)\cos sx\,ds = f(x)$$

$$\sqrt{2/\pi}\int_0^\infty \sqrt{2/\pi}\,\frac{a}{a^2+s^2}\cos sx\,ds = e^{-ax}$$

$$\frac{2a}{\pi}\int_0^\infty \frac{1}{a^2+s^2}\cos sx\,ds = e^{-ax}.$$

$$\int_0^\infty \frac{\cos sx}{s^2+a^2}\, ds = \frac{\pi}{2a} e^{-ax}$$

Put $s = x$ & $x = \alpha$

$$\int_0^\infty \frac{\cos \alpha x}{x^2+a^2}\, dx = \frac{\pi}{2a} e^{-a\alpha}.$$

using parseval's Identity fourier casine transform.

$$\int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |F(x)|^2\, dx.$$

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(a^2+s^2)^2}\, ds = \int_0^\infty e^{-2ax}\, dx.$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(a^2+s^2)^2}\, ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= 1/2a.$$

Put $s = x$

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\, dx = \pi/4a^3$$

Q. ... termine sine and cosine transform of e^{-4x} and hen

Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx$

$f(x) = e^{-ax} \qquad g(x) = e^{-bx}$

$F(c)\left[e^{-ax}\right] = \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx$

$= \sqrt{2/\pi} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + \sin sx) \right]_0^\infty$

$F_c\left[e^{-ax}\right] = \sqrt{2/\pi} \left(\frac{a}{a^2+s^2} \right) = F_c(s)$

III ly

$F_c\left[e^{-bx}\right] = \sqrt{2/\pi} \left(\frac{b}{b^2+s^2} \right) = G_c(s)$

using parseval's Identity

$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty e^{-ax} e^{-bx} dx$

$\int_0^\infty \sqrt{2/\pi} \left[\frac{a}{a^2+s^2}\right] \sqrt{2/\pi} \left[\frac{b}{b^2+s^2}\right] ds = \int_0^\infty e^{-(a+b)x} dx$

$\frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \left[\frac{e^{-(a+b)x}}{-a+b} \right]_0^\infty$

$\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \frac{\pi}{2ab(a+b)}$

put $s = x$

$ds = dx$

$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2ab(a+b)}$

Find the Fourier sine transform of x/x^2+a^2 and Fourier cosine transform of $\frac{a}{x^2+a^2}$

$f(x) = e^{-ax}$

$$F_c\left[e^{-ax}\right] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]$$

$$= \sqrt{2/\pi}\left[\frac{a}{a^2+s^2}\right] = F_c(s)$$

Taking inverse FCT,

$$\sqrt{2/\pi}\int_0^\infty F_c(s)\cos sx\,ds = f(x)$$

$$\sqrt{2/\pi}\int_0^\infty \sqrt{2/\pi}\left(\frac{a}{a^2+s^2}\right)\cos sx\,ds = e^{-ax}$$

$$\sqrt{2/\pi}\int_0^\infty \frac{a}{a^2+s^2}\cos sx\,ds = \sqrt{\pi/2}\,e^{-ax}$$

put $s = x$ & $x = s$

$$\sqrt{2/\pi}\int_0^\infty \frac{a}{a^2+x^2}\cos sx\,dx = \sqrt{\pi/2}\,e^{-as}$$

$$F_c\left[\frac{a}{a^2+x^2}\right] = \sqrt{\pi/2}\,e^{-as}$$

$f(x) = e^{-ax}$

$$F(s)\cdot\left[e^{-ax}\right] = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\sin sx + s\cos sx\right)\right]_0^\infty$$

$$= \sqrt{2/\pi}\left[\frac{s}{a^2+s^2}\right] = F_s(s)$$

Taking inverse F_sT

$$\sqrt{2/\pi}\int_0^\infty F_s(s)\sin sx\,ds = f(x)$$

$$\sqrt{2/\pi}\int_0^\infty \sqrt{2/\pi}\left(\frac{s}{a^2+s^2}\right)\sin sx\,ds = e^{-ax}$$

$$\sqrt{2/\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)\sin sx\,ds = \sqrt{2/\pi}\,e^{-ax}$$

put $s = x$ and $x = s$

$$\sqrt{2/\pi}\int_0^\infty \frac{x}{a^2+x^2}\sin sx\,dx = \sqrt{2/\pi}\,e^{-as}$$

$$F_s\left[\frac{x}{a^2+x^2}\right] = \sqrt{\pi/2}\,e^{-as}$$

Find the Fourier cosine-transform of $f(x) = \begin{cases} 1-x^2 & 0<x<1 \\ 0 & \text{else where} \end{cases}$

Hence deduce the value of $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx$ also

find $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\, dx$.

Given.

$$f(x) = \begin{cases} 1-x^2 & 0<x<1 \\ 0 & \text{else where} \end{cases}$$

$$F_c[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{2/\pi} \int_0^1 (1-x^2) \cos sx\, dx.$$

$$= \sqrt{2/\pi}\left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$= \sqrt{2/\pi}\left[\frac{-2\cos s}{s^2} + \frac{2\sin s}{s^3}\right]$$

$$= \sqrt{2/\pi}\left[\frac{2\sin s - 2s\cos s}{s^3}\right]$$

$$= 2\sqrt{2/\pi}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

Taking inverse cosine transform

$$\sqrt{2/\pi}\int_0^\infty F_c(s)\cos sx\, dx = f(x)$$

$$\sqrt{2/\pi}\int_0^\infty 2\sqrt{2/\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds = 1-x^2$$

put $x=0$

$$\frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = 1$$

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \pi/4$$

put $s = x$

$$\int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx = \pi/4$$

Using Parsval's Identity

$$\int_0^\infty |F(x)|^2 \, dx = \int_0^\infty |F_c(s)|^2 \, ds$$

$$\int_0^\infty (1-x^2)\, dx = 2/\pi \; 4 \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds$$

$$\int_0^\infty (1 - 2x^2 + x^4)\, dx = \frac{8}{\pi} \int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} \, ds$$

$$1 - 2/3 + 1/5 = 8/\pi \int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} \, ds$$

$$\frac{\cancel{8}}{15} = \frac{\cancel{8}}{\pi} \int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} \, ds$$

put $s = x$

$$\int_0^\infty \frac{(\sin x - x\cos x)^2}{x^6} \, dx = \pi/15$$

PROPERTIES

11. Find the Fourier sine transform of $F_s[f(x)\cos ax]$

Proof

$$F_s[f(x)\cos ax] = \sqrt{2/\pi}\int_0^\infty f(x)\cos ax \sin sx \, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\sin sx \cos ax \, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\left[\frac{\sin(s+a)x+\sin(s-a)x}{2}\right]dx$$

$$= \frac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin(s+a)x \, dx + \sqrt{2/\pi}\int_0^\infty f(x)\sin(s-a)x \, dx\right]$$

$$F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(s-a) + F_s(s+a)\right]$$

2. Find the sine transform of $f(x)$ sinax

$$F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

$$F_s[f(x)\sin ax] = \sqrt{2/\pi}\int_0^\infty f(x)\sin ax \sin sx\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\sin sx \sin ax\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\left[\frac{\cos(s-a)x - \cos(s+a)x}{2}\right] dx$$

$$= \frac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x)\cos(s+a)x - \sqrt{2/\pi}\int_0^\infty f(x)\cos(s-a)x\, dx\right.$$

$$F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

Find the fourier cosine transform of $f(x)\sin ax$ =

$\cancel{\frac{1}{2}[F_s(s-a) - F_s(s+a)]}$

$$F_c[f(x)\sin ax] = \sqrt{2/\pi}\int_0^\infty f(x)\sin ax \cos sx\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\left[\frac{\sin(s+a)x + \sin(s+a)x}{2}\right] dx$$

$$= \frac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin(s+a)x + \sqrt{2/\pi}\int_0^\infty f(x)\sin(s+a)x\right]$$

$$F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s+a)]$$

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14. Find the Fourier cosine transform of

$$F_c[f(x)\cos ax] = \tfrac{1}{2}[F_c \sin(s-a) + F_c \sin(s+a)]$$

$$F_c[f(x)\cos ax] = \sqrt{2/\pi}\int_0^\infty f(x)\cos ax \cos sx\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\cos sx \cos ax\, dx.$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\left[\frac{\cos(s-a)x + \cos(s+a)x}{2}\right] dx$$

$$= \tfrac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x)\cos(s-a)x\, dx + \sqrt{2/\pi}\int_0^\infty f(x)\cos(s+a)x\, dx\right]$$

$$F_c[f(x)\cos ax] = \tfrac{1}{2}[F_c(s-a) + F_c(s+a)]$$

15. Find the Fourier cosine transform of $F_c[x f(x)]$

$$F_s[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\, dx$$

Diff. w. r. to s once.

$$\frac{d}{ds}F_s[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\; x\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty x f(x)\cos sx\, dx$$

$$= F_c[x f(x)]$$

$$F_c[x f(x)] = d/ds\; F_s[f(x)]$$

Find the fourier cosine transform of $e^{-x^2/2}$ and the fourier sine transform of $x e^{-x^2/2}$

$$F_c\left[e^{-x^2/2}\right] = \sqrt{2/\pi}\int_0^\infty e^{-x^2/2}\cos sx\,dx$$

$$= \sqrt{2/\pi}\ \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2/2}\cos sx\,dx$$

$$= \frac{1}{2}\sqrt{2/\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\,R.P\,(e^{isx})\,dx$$

$$= R.P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2/2 - isx)}\,dx$$

$$= R.P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/\sqrt{2} - is/\sqrt{2})^2}\,e^{-s^2/2}\,dx$$

$$= e^{-s^2/2}\ R.P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/\sqrt{2} - is/\sqrt{2})^2}\,dx$$

Put $x/\sqrt{2} - is/\sqrt{2} = t$

$$\frac{dx}{\sqrt{2}} = dt$$

$$dx = \sqrt{2}\,dt$$

$$= e^{-s^2/2}\ R.P\left[\frac{1}{\sqrt{\cancel{2}\pi}}\int_{-\infty}^{\infty}\left[e^{-t^2}\cancel{\sqrt{2}}\,dt\right]\right]$$

$$= e^{-s^2/2}\ R.P\left[\frac{1}{\sqrt{\pi}}\sqrt{\pi}\right]$$

$$F_c\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

Here $e^{-x^2/2}$ is self

under cosine transform

$$F_s\left[x\, f(x)\right] = -d/ds\ F_c\left[f(x)\right]$$

$$F_s\left[x e^{-x^2/2}\right] = -d/ds\ F_c\left[e^{-x^2/2}\right]$$

$$= -d/ds\ e^{-s^2/2}$$

$$= -e^{-s^2/2}\left(-\cancel{2}s/\cancel{2}\right)$$

$$F_s\left[x\, e^{-x^2/2}\right] = s\, e^{-s^2/2}$$

$x\, e^{-x^2/2}$ is self reciprocal under fourier sine transform.

Find fourier cosine transform of $f(x) = e^{-x^2a^2}$ and hence fourier cosine transform of $e^{-x^2/2}$ and fourier sine transform of $x \cdot e^{-x^2/2}$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{2/\pi}\int_0^\infty e^{-a^2x^2}\cos sx\ dx.$$

$$= 1/2\sqrt{2/\pi}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\ dx$$

$$= 1/2\sqrt{2/\pi}\int_{-\infty}^{\infty} e^{-a^2x^2}\ R.p\left(e^{isx}\right) dx.$$

$$= R.P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$$= R P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax - is/2a)^2} e^{-s^2/4a^2} dx$$

$$= e^{-s^2/4a^2} \quad R.P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(ax - is/2a)^2}$$

Put $ax - is/2a = t$

$a \cdot dx = dt$

$dx = dt/a$

$$= e^{-s^2/4a^2} \, R.P \left[\frac{1}{\sqrt{2\pi}}\right] \int_{-\infty}^{\infty} e^{-t^2} dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \quad R.P \left[\frac{1}{\sqrt{2\cancel{\pi}}} \sqrt{\cancel{\pi}}\right]$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2}\, a}$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{e^{-s^2/4a^2}}{\sqrt{2}\, a.}$$

Put $a = \frac{1}{\sqrt{2}}$ in ①

$$F_c\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

$$F_s\left[x \, f(x)\right] = -d/ds \quad F_c\left[f(x)\right]$$

$$F_s\left[x e^{-x^2/2}\right] = -d/ds\ F_c\left[e^{-x^2/2}\right]$$

$$= -d/ds\left[e^{-s^2/2}\right]$$

$$F_s\left[x e^{-x^2/2}\right] = s e^{-s^2/2}$$

$x e^{-x^2/2}$ is self reciprocal under fourier sine transform.

Find fourier cosine transform of e^{-x^2}

$$F_c[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-x^2}\right] = \sqrt{2/\pi}\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$= 1/2\ \sqrt{2/\pi}\int_{-\infty}^\infty e^{-x^2}\ R\cdot P\,(e^{isx})\,dx$$

$$= R\cdot P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(x^2-isx)}\,dx$$

$$= R\cdot P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(x-is/\sqrt{2})^2}\ e^{-s^2/4}\,dx$$

$$= e^{-s^2/4}\ R\cdot P\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(x-is/2)^2}\,dx$$

$$x - is/2 = t$$

$$dx = dt$$

$$= e^{-s^2/4} \; R\cdot P \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$= e^{-s^2/4} \; R\cdot P \; \frac{1}{\sqrt{2\cancel{\pi}}} \; \sqrt{\cancel{\pi}}$$

$$= e^{-s^2/4} \; R\cdot P \; \frac{1}{\sqrt{2}}$$

$$F_c \left[e^{-x^2} \right] = \frac{e^{-s^2/4}}{\sqrt{2}}$$

UNIT – V

Z – TRANSFORM

ONE – SIDED Z – TRANSFORM (OR) UNILATERAL Z – TRANSFORM

The one sided z – transform of a causal sequence $f(n)$ is defined as

$$\boxed{z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n} = \bar{f}(z) \text{ (or) } F(z)}$$

TWO SIDED Z – TRANSFORM (OR) BILATERAL Z – TRANSFORM

The two sided z – transform of a non causal sequence $f(n)$ is called as

$$\boxed{z[f(n)] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} = \bar{f}(z) \text{ (or) } F(z)}$$

Z – TRANSFORM OF TIME SEQUENCE

If $f(t)$ is defined in its sampled values $0, T, 2T \ldots$ then the z – transform of $f(t)$ is defined as

$$\boxed{z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n} = \bar{f}(z) \text{ or } F(z)}$$

INVERSE Z – TRANSFORM

If $F(z)$ is a z – transform of any sequence $f(n)$, then the inverse z – transform of z-transform function $F(z)$ is defined as

$$\boxed{z^{-1}[F(z)] = f(n)}$$

(i) Time shifting property

(i) $Z[f(n-n_0)] = z^{-n_0}\bar{f}(z)$ (or) $z^{-n_0} F(z)$

Second shifting (ii) $Z[f(t+T)] = z[\bar{f}(z) - f(0)]$

Property (i):

$$Z[f(n-n_0)] = \sum_{n=0}^{\infty} f(n-n_0) z^{-n}$$

Put $n-n_0 = m$
$n = m+n_0$
Where $n=0$
$m = -n_0$
$n=\infty$, $m = \infty$

$$= \sum_{m=-n_0}^{\infty} f(m) z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-n_0}^{\infty} f(m) z^{-m} = z^{-n_0} \sum_{m=0}^{\infty} f(m) z^{-m}$$

[$\therefore f(m)$ is casual

$$\boxed{Z[f(n-n_0)] = z^{-n_0}\bar{f}(z)}$$

(ii) $$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f[(n+1)T] z^{-n}$$

Put
$n+1 = m$
$n = m-1$
When $n=0$
$m=1$
$n=\infty$, $m = \infty$

$$= \sum_{m=1}^{\infty} f(mT) z^{-(m-1)}$$

$$= z \sum_{m=1}^{\infty} f(mT) z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} f(mT) z^{-m} - f(0)\right]$$

$$\boxed{Z[f(t+T)] = z[\bar{f}(z) - f(0)]}$$

Time reversal property for bilateral z-transform

8) If $Z[f(n)] = \bar{f}(z)$ then $Z[f(-n)] = \bar{f}(1/z)$

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$$Z[f(-n)] = \sum_{n=-\infty}^{\infty} f(-n) z^{-n}$$

$-n = m$
$n = -m$
when
$n = -\infty$
$m = \infty$
$n = \infty$
$m = -\infty$

$$= \sum_{m=\infty}^{-\infty} f(m) z^{m}$$

$$= \sum_{n=-\infty}^{\infty} f(m) (1/z)^{-m}$$

$$\boxed{Z[f(-n)] = \bar{f}(1/z)}$$

3) Differentiation in Z-transform

(i) $Z[n f(n)] = -z \frac{d}{dz} [\bar{f}(z)]$

(ii) $Z[n f(t)] = -z \frac{d}{dz} [\bar{f}(z)]$

Proof (i)

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Diff par. w. r. to 'z'

$$\frac{d}{dz} \bar{f}(z) = -1/z \sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$= -1/z \; Z[n f(n)]$$

$$Z[n f(n)] = -z \frac{d}{dz} [\bar{f}(z)]$$

Proof (ii)

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

Diff p. w. r. to 'z'

$$\frac{d \bar{f}(z)}{dz} = \sum_{n=0}^{\infty} f(nT) (-n) z^{-n-1}$$

$$\frac{d}{dz}\bar{f}(z) = -1/z \sum_{n=0}^{\infty} n f(nT) z^{-n}$$

$$= -1/z \; z[n f(t)]$$

$$\boxed{z[n f(t)] = -z \frac{d}{dz}[F(z)]}$$

Change of scale property First Shifting Theorem

(i) If $z[f(n)] = \bar{f}(z)$ then $z[a^n f(n)] = \bar{f}(z/a)$

(ii) If $z[f(t)] = \bar{f}(z)$ then $z[a^n f(t)] = \bar{f}(z/a)$

Proof (i)

$$z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$\left\{\frac{z^{-n}}{a^{-n}} = z/\right.$

$$= \sum_{n=0}^{\infty} f(n) (z/a)^{-n}$$

$$= \bar{f}(z/a)$$

Proof (ii)

$$z[a^n f(t)] = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (z/a)^{-n} \qquad \left[\frac{z^{-n}}{a^{-n}} = (z/a)^{-n}\right.$$

$$= \bar{f}(z/a)$$

(B) i

If $z[f(t) = F(z)$ then $z[e^{at} f(t)] = \bar{f}(z e^{-at})$

$$z[e^{at} f(t)] = \sum_{n=0}^{\infty} e^{anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT})^n f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT})^{-n} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} (z e^{-aT})^{-n} f(nT)$$

$$\boxed{Z[e^{at} f(t)] = \bar{f}(ze^{-aT})}$$

(iv) If $Z[f(t)] = \bar{f}(z)$ then $Z[e^{-at} f(t)] = \bar{f}(ze^{aT})$

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT})^{-n} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(ze^{aT})^{-n}$$

$$\boxed{Z[e^{-at} f(t)] = \bar{f}(ze^{aT})}$$

Initial Value Theorem:

(i) If $Z[f(n)] = \bar{f}(z)$ then $\lim_{z\to\infty} \bar{f}(z) = \lim_{n\to 0} f(n)$

(ii) If $Z[f(t)] = \bar{f}(z)$ then $\lim_{z\to\infty} \bar{f}(z) = \lim_{t\to 0} f(t)$

Proof (i) $\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$= f(0) + \frac{f(1)}{z^1} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \cdots$$

$$\lim_{z\to\infty} \bar{f}(z) = f(0)$$

$$\lim_{z\to\infty} \bar{f}(z) = \lim_{n\to 0} f(n)$$

Proof (ii)

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \frac{f(3T)}{z^3} + \cdots$$

$$\lim_{z \to \infty} \bar{f}(z) = f(0)$$

$$\boxed{\lim_{z \to \infty} \bar{f}(z) = \lim_{t \to 0} f(t)}$$

Final value theorem

(i) If $z[f(n)] = \bar{f}(z)$ then $\lim_{z \to 1} (z-1)\bar{f}(z) = \lim_{n \to \infty} f$

(ii) If $z[f(t)] = \bar{f}(z)$ then $\lim_{z \to 1} (z-1)\bar{f}(z) = \lim_{t \to \infty} f(t)$

Proof(i)

$$z[f(n+1)] = \sum_{n=0}^{\infty} f[(n+1)] z^{-n}$$

$n+1 = m$

$n = m-1$

when $n=0$, $m=1$

$n=\infty$ $m=\infty$

$$= \sum_{m=1}^{\infty} f(m) z^{-(m-1)}$$

$$= z \sum_{m=1}^{\infty} f(m) z^{-m}$$

$$= z \left[\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) \right]$$

$$\boxed{z[f(n+1)] = z[\bar{f}(z) - f(0)]}$$

$$z[\bar{f}(z) - f(0)] = z[f(n+1)]$$

$$z\bar{f}(z) - z f(0) - \bar{f}(z) = z[f(n+1)] - z[f(n)]$$

$$(z-1)\bar{f}(z) - z f(0) = z[f(n+1) - f(n)]$$

$$(z-1)\bar{f}(z) - z f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$\lim_{z \to 1} (z-1)\bar{f}(z) - f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]$$

$$= f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \cdots$$

$$= \lim_{n \to \infty} [f(n+1) - f(0)]$$

$$\lim_{z \to 1} (z-1)\bar{f}(z) - \cancel{f(0)} = \lim_{n \to \infty} f(n+1) - \cancel{f(0)}$$

$$\lim_{z \to 1} (z-1)\bar{f}(z) = f(\infty)$$

$$= \lim_{n \to \infty} f(n)$$

$$\boxed{\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1)\bar{f}(z)}$$

Proof (ii)

$$z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n}$$

Put $n+1 = m$

$n = m-1$

When $n=0, m=1$

$n = \infty, m = \infty$

$$= \sum_{n=1}^{\infty} [f(n+1)T]\, z^{-n}$$

$$= \sum_{m=1}^{\infty} f(mT)\, z^{-(m-1)}$$

$$= z\left[\sum_{n=0}^{\infty} f(mT)\, z^{-m} - f(0)\right]$$

$$\boxed{z[f(t+T)] = z[\bar{f}(z) - f(0)]}$$

$$z[\bar{f}(z) - f(0)] = z[f(t+T)]$$

$$z\bar{f}(z) - zf(0) - \bar{f}(z) = z[f(t+T)] - z[f(t)]$$

$$(z-1)\bar{f}(z) - zf(0) = z[f(t+T) - f(t)]$$

$$(z-1)\bar{f}(z) - zf(0) = \sum_{n=0}^{\infty} [f(n+1)T - f(nT)]\, z^{-n}.$$

$$\lim_{z\to 1} (z-1)\, \bar{f}(z) - f(0) = \sum_{n=0}^{\infty} [f(n+1)T - f(nT)]$$

$$= f(T) - f(0) + f(2T) - f(T) + f(3T) - f(2T) \ldots$$

$$= \lim_{n\to\infty} [f(n+1)T - f(0)]$$

$$\lim_{z\to 1} (z-1)\, \bar{f}(z) - f(0) = \lim_{n\to\infty} f[(n+1)T] - f(0)$$

$$\lim_{z\to 1} (z-1)\, \bar{f}(z) = f(\infty)$$

$$= \lim_{t\to\infty} f(t)$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\, \bar{f}(z)}$$

1. Z - transform of some basic functions

(i) If the unit impulse sequence.

$$\delta(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

$$\delta(n-1) = \begin{cases} 1 & \text{if} \\ 0 & \end{cases}$$

$$\delta(n-2) = \begin{cases} 1 & n \\ 0 & n \end{cases}$$

$$z[\delta(n)] = 1$$

$$z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}.$$

$z[\delta(n)] =$

$z[\delta(n-1)]$

$z[\delta(n-2)]$

2. Find the z - transform of unit step sequence

$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$z[u(n)] = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + 1/z + 1/z^2 + \dots$$

$$= (1 - 1/z)^{-1}$$

$$= \left[\frac{z-1}{z}\right]^{-1} = \frac{z}{z-1}$$

3. Find the z- transform of k

$$z[k] = \sum_{n=0}^{\infty} k\, z^{-n}$$

$$= k\left[1 + 1/z + 1/z^2 + \dots\right]$$

$$z[k] = \frac{kz}{z-1}$$

Result:

$$z[1] = \frac{z}{z-1}$$

$$z[2] = \frac{2z}{z-1}$$

$$z[3] = \frac{3z}{z-1}$$

4. Find the z- transform of a^n.

$$z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a/z)^n$$

$$= 1 + a/z + (a/z)^2 + \dots$$

$$= (1 - a/z)^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$\boxed{Z[a^n] = \frac{z}{z-a}}$$

5. Find the z-transform of e^{at}

$$Z[e^{at}] = Z[e^{anT}]$$

$$= Z[(e^{aT})^n]$$

$$= \frac{z}{z-e^{aT}} \qquad \left[\because Z[a^n] = \frac{z}{z-a}\right]$$

(or)

$$Z[e^{aT}] = \sum_{n=0}^{\infty} e^{anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} \cancel{\frac{z^{-n}}{e^{-anT}}} = \sum_{n=0}^{\infty} \frac{(e^{aT})^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \cancel{\left(\frac{z}{e^{aT}}\right)^{-n}} \quad \left(\frac{e^{aT}}{z}\right)^n$$

$$= 1 + \frac{e^{aT}}{z} + \left(\frac{e^{aT}}{z}\right)^2 + \cdots\cdots$$

$$= \left[1 - \frac{e^{aT}}{z}\right]^{-1}$$

$$= \left[\frac{z-e^{aT}}{z}\right]^{-1}$$

$$\boxed{Z[e^{at}] = \frac{z}{z-e^{aT}}}$$

6. Find the z-transform of e^{-at}

$$Z[e^{-at}] = Z[e^{-anT}]$$

$$= Z[(e^{aT})^{-n}]$$

$$z\left[e^{-aT}\right] = \frac{z}{z-e^{-aT}}$$

(or)

$$z\left[e^{-at}\right] = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} \frac{(e^{-aT})^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{e^{anT}} = \sum_{n=0}^{\infty}\left(\frac{e^{-aT}}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{z}{e^{-aT}}\right)^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{e^{-aT}}{z}\right)^n$$

$$= 1 + \frac{e^{-aT}}{z} + \left(\frac{e^{-aT}}{z}\right)^2 + \cdots\cdots$$

$$= \left[1 - \frac{e^{-aT}}{z}\right]^{-1}$$

$$= \left[\frac{z-e^{-aT}}{z}\right]^{-1}$$

$$\boxed{z\left[e^{-at}\right] = \left[\frac{z}{z-e^{-aT}}\right]}$$

7. Find the z-transform of n.

$$z[n] = \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= 1/z + \frac{2}{z^2} + 3/z^3 + \cdots\cdots$$

$$= 1/z\left[1 + \frac{2}{z} + 3/z^2 + \cdots\right]$$

$$= 1/z\left[1 - 1/z\right]^{-2}$$

$1+2x+3x^2+\cdots = (1-x)^{-2}$

$$= 1/z \left[\frac{z-1}{z}\right]^{-2}$$

$$= 1/\cancel{z} \left[\frac{z^{\cancel{2}}}{(z-1)^2}\right]$$

$$\boxed{z[n] = \frac{z}{(z-1)^2}}$$

8. From the z- transform of a^{n-1}

$$z[a^{n-1}] = z^{-1} z[a^n]$$

$$= z^{-1} \frac{z}{z-a}$$

$$\boxed{z[a^{n-1}] = \frac{1}{z-a}}$$

9. Find the z- transform of na^n

$$z[n\, f(n)] = -z \frac{d}{dz} \bar{f}(z)$$

$$z[n\, a^n] = -z \frac{d}{dz} \left[\frac{z}{z-a}\right]$$

$$= -z \left[\frac{(\cancel{z}-a) - z}{(z-a)^2}\right]$$

$$\boxed{z[na^n] = \frac{az}{(z-a)^2}}$$

10. Find the z- transform of n^2

$$z[n^2] = z[n\ \underline{n}]$$

$$= -z \frac{d}{dz} z[n]$$

$$= -z \frac{d}{dz} \left[\frac{z}{(z-1)^2}\right]$$

$$= -z \left[\frac{(z-1)^2 - z.\ 2(z-1)}{(z-1)^4}\right]$$

$$= -z(z-1)\left[\frac{(z-1)-2z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$\boxed{Z[n^2] = \frac{z(z+1)}{(z-1)^3}}$$

11. Find the z-transform of $[n(n-1)]$

$$Z[n(n-1)] = Z[n^2-n]$$

$$= Z[n^2] - Z[n]$$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$

$$= \frac{z^2+z-(z^2-z)}{(z-1)^3}$$

$$\boxed{Z[n(n-1)] = \frac{2z}{(z-1)^3}}$$

12. Find the z-transform of $[n(n+1)]$

$$Z[n(n+1)] = Z[n^2+n]$$

$$= Z[n^2] + Z[n]$$

$$= \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) + z(z-1)}{(z-1)^3}$$

$$= \frac{z^2+z\,(z^2-z)}{(z-1)^3}$$

$$\boxed{z\,[n\,(n+1)] = \frac{2z^2}{(z-1)^3}}$$

Formula:

$$1/z + 1/2z^2 + 1/3z^3 + \cdots = -\log(1 - 1/z)$$

(or)

$$x + x^2/2 + x^3/3 + \cdots = -\log(1-x)$$

13. Find the z-transform of $1/n$

$$z[1/n] = \sum_{n=1}^{\infty} 1/n\, z^{-n}$$

$$= 1/z + 1/2z^2 + 1/3z^3 + \cdots$$

$$= -\log(1 - 1/z)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{z(1/n) = \log\left(\frac{z}{z-1}\right)}$$

14. Find the z-transform of $1/n+1$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} 1/n+1\; z^{-n}$$

$$= 1 + 1/2z + 1/3z^2 + \cdots$$

$$= z\left[1/z + 1/2z^2 + 1/3z^3 + \cdots\right]$$

$$= z\left[-\log(1 - 1/z)\right]$$

$$= z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= z\left[\log\left(\frac{z-1}{z}\right)^{-1}\right]$$

$$\boxed{Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)}$$

15. Find the z-transform of $1/n-1$

$$Z\left[\frac{1}{n-1}\right] = \sum_{n=0}^{\infty}\frac{1}{n-1}z^{-n}$$

$$= -1 + 1/z^2 + \frac{1}{2z^3} + \frac{1}{3z^4} + \cdots$$

$$= -1 + 1/z\left[1/z + 1/2z^2 + 1/3z^3 + \cdots\right]$$

$$= -1 + 1/z\left[-\log(1 - 1/z)\right]$$

$$= -1 + 1/z\log\left(\frac{z}{z-1}\right)$$

$$\boxed{Z\left[\frac{1}{n-1}\right] = 1/z\log\left(\frac{z}{z-1}\right) - 1}$$

16. Find the z-transform of $z\left[\frac{a^n}{n!}\right]$ and $\left[\frac{1}{n!}\right]$

$$Z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n}$$

$$= 1 + a/z\,\frac{1}{1!} + (a/z)^2\,1/2! + \cdots$$

$$= e^{a/z}$$

$$\boxed{Z\left[\frac{a^n}{n!}\right] = e^{a/z}}$$

(ii) $z\left[\dfrac{a^n}{n!}\right] = e^{a/z}$

put $a=1$

$$\boxed{z\left[\dfrac{1}{n!}\right] = e^{1/z}}$$

17. Find the z- transform of $\dfrac{1}{n+2}$

$$z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \tfrac{1}{2} + \tfrac{1}{3z} + \tfrac{1}{4z^2} + \tfrac{1}{5z^3} + \cdots$$

$$= z^2\left[\frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \right]$$

$$= z^2\left[\frac{1}{2z^2} + \tfrac{1}{3z^3} + \tfrac{1}{4z^4} + \cdots\right]$$

$$= z^2\left[\left(\tfrac{1}{z} + \tfrac{1}{2z^2} + \tfrac{1}{3z^3} + \tfrac{1}{4z^4} \cdots\right) - \tfrac{1}{z}\right]$$

$$= z^2\left[-\log\left(1-\tfrac{1}{z}\right) - \tfrac{1}{z}\right]$$

$$= z^2\left[\log\frac{z}{z-1} - \tfrac{1}{z}\right]$$

$$\boxed{z\left[\frac{1}{n+2}\right] = z^2 \log\left(\frac{z}{z-1}\right) - z}$$

19. Find the z- transform of

$z[r^n \cos n\theta]$ $\quad$ $z[\cos n\theta]$ $\quad$ $z[\cos n\pi/2]$

$z[r^n \sin n\theta]$ $\quad$ $z[\sin n\theta]$ $\quad$ $z[\sin n\pi/2]$

$$z[a^n] = \frac{z}{z-a}$$

Put $a = re^{i\theta}$.

$$z\left[r^n e^{in\theta}\right] = \frac{z}{z - re^{i\theta}}$$

$$z\left[r^n(\cos n\theta + i\sin n\theta)\right] = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$z\left[r^n\cos n\theta + i r^n \sin n\theta\right] = \frac{z}{(z - r\cos\theta) - ir\sin\theta}$$

$$z\left[r^n\cos n\theta\right] + iz\left[r^n \sin n\theta\right] = \frac{z\left[(z - r\cos\theta) + ir\sin\theta\right]}{z^2 - 2zr\cos\theta + r^2}$$

$$z\left[r^n\cos n\theta\right] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

$$z\left[r^n\sin n\theta\right] = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

put $r = 1$

$$z\left[\cos n\theta\right] = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

put $\theta = \pi/2$

$$z\left[\cos n\pi/2\right] = \frac{z^2}{z^2 + 1}$$

$$z\left[\sin n\pi/2\right] = \frac{z}{z^2 + 1}$$

Inverse z-transform.

(i) $z^{-1}\left[\frac{kz}{z-1}\right] = k$

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$$z^{-1}\left[\frac{z}{z-1}\right] = 1$$

$$z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

$$z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

$$z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n$$

$$z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos n\pi/2$$

$$z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = n^2.$$

$$z^{-1}\left[\frac{1}{z-a}\right] = a^{n-1}$$

$$z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$$

$$z^{-1}\left[\frac{z}{z^2+1}\right] = \sin n\pi/2$$

$$z^{-1}\left[\frac{az}{z^2+a^2}\right] = a^n \sin n\pi/2$$

$$z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = a^n \cos n\pi/2$$

Convolution ~~theorem~~ Definition

If $f(n)$ & $g(n)$ are causual sequences then convolution of $f(n)$ & $g(n)$ is defined as

$$f(n) * g(n) = \sum_{r=0}^{n} f(r)\, g(n-r)$$

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CONVOLUTION THEOREM

If inverse z-transform of $f(z)$ & $g(z)$ are z-transform of sequences $f(n)$ & $g(n)$ respectively

$$z^{-1}[f(z)\,g(z)] = z^{-1}[f(z)] * z^{-1}[g(z)]$$

Note: $\boxed{\sum_{r=0}^{n} a^r = \frac{1-a^{n+1}}{1-a}}$

Find $z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right]$

$$z^{-1}[f(z)g(z)] = z^{-1}[f(z)] * z^{-1}[g(z)]$$

$$z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right] = z^{-1}\left[\frac{z}{z-1/2}\right] * z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$= z^{-1}\left[\frac{z}{z-1/2} \cdot \frac{z}{z-1/4}\right] = (1/2)^n * (1/4)^n$$

$$= \sum_{r=0}^{n} (1/2)^r (1/4)^{n-r}$$

$$= (1/4)^n \sum_{r=0}^{n} (2)^r$$

$$= (1/4)^n \left[\frac{1-(2)^{n+1}}{1-2}\right]$$

$$= (1/4)^n \left[(2)^{n+1} - 1\right]$$

$$= (1/4)^n \left[2^n \cdot 2 - 1\right]$$

$$= 2^n \cdot 2\,(1/4)^n - (1/4)^n$$

$$= \frac{2}{2^n} - \frac{1}{4^n} = 2^{1-n} - 1/4^n$$

$$= \frac{1}{2^{n-1}} - 1/4^n$$

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Using convolution theorem $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{r=0}^{n} a^r \; b^{n-r}$$

$$= b^n \sum_{r=0}^{n} a^r \; b^{-r}$$

$$= b^n \left[\frac{1-(a/b)^{n+1}}{1-a/b}\right]$$

$$= b^n \left[\frac{1-\frac{a^{n+1}}{b^{n+1}}}{(b-a)/b}\right]$$

$$= b^{\cancel{n}+1} \left[\frac{\frac{b^{n+1}-a^{n+1}}{b^{\cancel{n+1}}}}{b-a}\right]$$

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}.$$

Find $z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]$

$$z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = z^{-1}\left[\frac{z}{(z+a)}\right] * z^{-1}\left[\frac{z}{(z+b)}\right]$$

$$= (-a)^n * (-b)^n$$

$$= \sum_{r=0}^{n} (-a)^r \; (-b)^{n-r}$$

$$= (-b)^n \sum_{r=0}^{n} (-a)^r \; (-b)^{-r}$$

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$$= (-b)^n \sum_{r=0}^{n} (-a)^r \, (-b)^{-r}$$

$$= (-b)^n \sum_{r=0}^{n} (-a/b)^r$$

$$= (-b)^n \sum_{r=0}^{n} (a/b)^r$$

$$= (-b)^n \left[\frac{1-(a/b)^{n+1}}{1 - a/b} \right]$$

$$= (-b)^n \left[\frac{1 - \frac{a^{n+1}}{b^{n+1}}}{(b-a)/b} \right]$$

$$= b^n (-1)^n \left[\frac{b^{n+1} - a^{n+1}}{b^{n+1}(b-a)} \right]$$

$$= (-1)^n \left[\frac{b^{n+1} - a^{n+1}}{b-a} \right]$$

$$z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = (-1)^n \left[\frac{b^{n+1} - a^{n+1}}{b-a} \right]$$

Using convolution theorem to find

$$z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$$

$$z^{-1}\left[\bar{f}(z) \cdot \bar{g}(z)\right] = z^{-1}\left[\bar{f}(z)\right] * z^{-1}\left[\bar{g}(z)\right]$$

$$z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = z^{-1}\left[\frac{\cancel{8}z^2}{\cancel{2}(z-1/2)\,\cancel{4}(z+1/4)}\right]$$

$$= z^{-1}\left[\frac{z}{(z-1/2)} \quad \frac{z}{z+1/4}\right]$$

$$= (1/2)^n * (-1/4)^n$$

$$= \sum_{r=0}^{n} (1/2)^r \, (-1/4)^{n-r}$$

$$= (-1/4)^n \sum_{r=0}^{n} (1/2)^r \, (-1/4)^{-r}$$

$$= (-1/4)^n \sum_{r=0}^{n} (1/2)^r (-4)^r$$

$$= (-1/4)^n \sum_{r=0}^{n} (-2)^r$$

$$= (-1/4)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right]$$

$$= (-1/4)^n \cdot 1/3 \left[1 + 2(-2)^n\right]$$

$$z^{-1} \frac{8z^2}{(2z+1)(4z+1)} = 1/3 \left(-1/4\right)^n + 2/3 (1/2)^n.$$

use convolution theorem of find $z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$

$$z^{-1}\left[\bar{f}(z)\ \bar{g}(z)\right] = z^{-1}\left[\bar{f}(z)\right] * z^{-1}\left[\bar{g}(z)\right]$$

$$z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = z^{-1}\left[\frac{z}{z+a}\right] * z^{-1}\left[\frac{z}{z+a}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{r=0}^{n} (-a)^r \, (-a)^{n-r}$$

$$= \sum_{r=0}^{n} (-a)^n$$

$$= (-a)^n \sum_{r=0}^{n} 1.$$

$$z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = (-a)^n\,(n+1)$$

using convolution theorem find $z^{-1}\left[\frac{z^2}{(z-1)\,(z-3)}\right]$

$$z^{-1}\left[\bar{f}(z) * \bar{g}(z)\right] = z^{-1}\left[\bar{f}(z)\right] * z^{-1}\left[\bar{g}(z)\right]$$

$$= z^{-1}\left[\frac{z}{z-1}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (1)^n * (3)^n$$

$$= \sum_{r=0}^{n} (1)^r\,(3)^{n-r}$$

$$= (3)^n \sum_{r=0}^{n} (1)^r\,(3)^{-r}$$

$$= (3)^n \sum_{r=0}^{n} (1/3)^r$$

$$= 3^n\left[\frac{1-(1/3)^{n+1}}{1-1/3}\right]$$

$$= 3^n\left[3/2\left(1-(1/3)^{n+1}\right)\right]$$

$$= 3/2\;3^n\left[1-(1/3)^{n+1}\right]$$

$$= \frac{3^{n+1}}{2}\left[1-(1/3)^{n+1}\right]$$

$$= \frac{3^{n+1}}{2} - \frac{\cancel{3^{n+1}}}{\cancel{3^{n+1}}}\;1/2$$

$$= \frac{3^{n+1}-1}{2}$$

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = 1/2\left[3^{n+1}-1\right]$$

Problem based on inverse z-transform using Partial fractions.

Find the $z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$ using partial fractions.

$$z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$$

$$\bar{f}(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\frac{\bar{f}(z)}{z} = \frac{z^2}{(z-1)^2(z-2)}$$

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$z^2 = A(z-1)^{\cancel{2}}(z-2) + B(z-2) + C(z-1)^2$$

put $z = 1$

$1 = -B$

$\boxed{B = -1}$

put $z = 2$

$\boxed{4 = C}$

put $z = 0$

$0 = 2A - 2B + C$

$0 = 2A + 2 + 4$

$2A = -6$

$\boxed{A = -3}$

$$\frac{\bar{f}(z)}{z} = \frac{-3}{z-1} - \frac{1}{(z-1)^2} + \frac{4}{z-2}$$

$$\bar{f}(z) = -3\left[\frac{z}{z-1}\right] - \left[\frac{z}{(z-1)^2}\right] + 4\left[\frac{z}{z-2}\right]$$

$$Z^{-1}[F(z)] = -3Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -3(1)^n - n(1)^n + 4\, 2^n$$

$$= 4(2)^n - (n+3)$$

Find $z^{-1}\left[\dfrac{z(z^2+z+2)}{(z+1)(z-1)^2}\right]$ using partial fractions.

$$F(z) = \frac{z(z^2+z+2)}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{z^2+z+2}{(z+1)(z-1)^2}$$

$$\frac{z^2+z+2}{(z+1)(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1}$$

$$z^2+z+2 = A(z-1)(z+1) + B(z+1) + C(z-1)^2$$

put $z = 1$

$4 = 2B$

$\boxed{B = 2}$

put $z = 1$

$2 = 4C$

$\boxed{C = 1/2}$

put $z = 0$

$2 = -A + B + C$

$\cancel{2} = -A + \cancel{2} + 1/2$

$+A = 1/2$

$\boxed{A = 1/2}$

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$$\frac{\bar{F}(z)}{z} = \frac{2}{(z-1)^2} + \frac{1}{2}\frac{1}{z+1} + \frac{1}{2}\frac{1}{z-1}$$

$$\bar{F}(z) = 2\frac{z}{(z-1)^2} + \frac{1}{2}\frac{z}{z+1} + \frac{1}{2}\frac{z}{z-1}$$

$$z^{-1}[\bar{F}(z)] = 2z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{2}z^{-1}\left[\frac{z}{z+1}\right] + \frac{1}{2}z^{-1}\left(\frac{z}{z-1}\right)$$

$$z^{-1}[F(z)] = 2n + \frac{(-1)^n}{2} + \frac{1}{2}$$

Find the $z^{-1}\left[\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right]$ using partial Fractions

$$z^{-1}\left[\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right]$$

$$= \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$3z^2-18z+26 = A(z-3)(z-4) + B(z-\cancel{3}2)(z-4) + C(z-2)(z-3)$$

Put $z=3$

$-B = 27-54+26$

$\boxed{B=1}$

Put $z=2$

$A(-1)(-2) = 12-36+26$

$\boxed{A=1}$

Put

$z=4$

$2C = 2$

$\boxed{C=1}$

$$\frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$$

$$z^{-1}\left[\frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}\right] = z^{-1}\left[\frac{1}{z-2}\right] + z^{-1}\left[\frac{1}{z-3}\right] + z^{-1}\left[\frac{1}{z-4}\right]$$

$$= 2^{n-1} + 3^{n-1} + 4^{n-1}$$

$$z^{-1}\left[\frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}\right] = \frac{2^n}{2} + \frac{3^n}{3} + \frac{4^n}{4}$$

Find the $z^{-1}\left[\frac{4z^3}{(2z-1)^2(z-1)}\right]$ using partial fractions

$$F(z) = \frac{4z^3}{(2z-1)^2(z-1)}$$

$$\frac{F(z)}{z} = \frac{4z^2}{(2z-1)^2(z-1)}$$

$$4z^2 = \frac{A}{2z-1} + \frac{B}{(2z-1)^2} + \frac{C}{z-1}$$

$$4z^2 = A(2z-1)(z-1) + B(z-1) + C(2z-1)^2$$

Put $z = 1/2$	Put $z = 1$
$-1/2\, B = 1$	$C = 4$
$\boxed{B = -2}$	

Equating co-efficient of z^2

$$4 = 4C + 2A.$$

$A = -6$

$$\frac{4z^2}{(2z-1)^2(z-1)} = \frac{-6}{2z-1} - \frac{2}{(2z-1)^2} + \frac{4}{(z-1)}$$

$$\frac{4z^3}{(2z-1)^2(z-1)} = \frac{-6z}{2z-1} - \frac{2z}{(2z-1)^2} + \frac{4z}{z-1}$$

$$\frac{4z^3}{(2z-1)^2(z-1)} = +2\left[\frac{-3z}{2z-1} + \frac{z}{(2z-1)^2} + \frac{2z}{z-1}\right]$$

$$z^{-1}\left[\frac{4z^3}{(2z-1)^2(z-1)}\right] = (-3)\, z^{-1}\left[\frac{z}{z-1/2}\right] - z^{-1}\left[\frac{\frac{1}{2}z}{(z-1/2)^2}\right]$$

$$+\, z^{-1}(4)\left[\frac{z}{(z-1)}\right]$$

$$= (-3)(1/2)^n - n(1/2)^n + 4(1)^n$$

$$= 4 - (1/2)^n\,[n+3]$$

Find $z^{-1}\left[\frac{z^2+z}{(z-1)(z^2+1)}\right]$ using partial Fractions.

$$F(z) = \frac{z^2+z}{(z-1)(z^2+1)} \qquad \frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2+1)}$$

$$\frac{(z+1)}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$z+1 = A(z^2+1) + (Bz+C)(z-1)$$

put $z=1$

$2 = 2A$

$\boxed{A=1}$

put $z=0$

$1 = A - C$

$\boxed{C=0}$

3.24
322

Put $z = 2$

$3 = 5A + (2B + C)$

$3 = 5 + 2B$

$2B = -2$

$\boxed{B = -1}$

$$\frac{F(z)}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$z^{-1} F(z) = z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$z^{-1}\left[\frac{z^2+z}{(z-1)(z^2+1)}\right] = 1 - \cos n\pi/2$$

Find the $z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$

$$F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

$$\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$+z = A(z^2+4) + (Bz+C)(z+2)$$

put $z = -2$

$-2 = 8A$

$A = -1/4$

put $z = 0$

$0 = 4A + 2C$

$0 = 4 \times -1/4 + 2C.$

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$2C = 1$

$$\boxed{C = 1/2}$$

Put $z = 1$

$1 = 5A + 3B + 3C$

$1 - 3/2 = -5/4 + 3B$

$-1/2 + 5/4 = 3B$

$$\boxed{B = 1/4}$$

$$\frac{F(z)}{z} = \frac{-1}{4(z+2)} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4}$$

$$F(z) = \frac{-z}{4\,z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}$$

$$z^{-1}[F(z)] = -\frac{1}{4} z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{\ldots}\right.$$

$$z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right] = -\frac{1}{4}(-2)^n + \frac{1}{4}(2)^n \cos n\pi/2 + \frac{1}{4}(2)^n \sin n\pi/2$$

$$z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right] = \frac{1}{4}\left[-(-2)^n + 2^n \cos n\pi/2 + 2^n \sin n\pi/2\right]$$

(Ans)

Cauchy's Residue theorem

$z^{-1}[\bar{f}(z)]$ = sum of residues of $z^{n-1}[\bar{f}(z)]$

Inverse z-transform using cauchy's residue theorem.

Residue formula :-

If $z=a$ is a simple pole

then $\underset{z=a}{\text{Res}} = \lim_{z \to a} (z-a)\, z^{n-1} \bar{f}(z)$

If $z=a$ is a double pole

then $\underset{z=a}{\text{Res}} = \lim_{z \to a} \frac{d}{dz} (z-a)^2 z^{n-1} \bar{f}(z)$

If $z=a$ is a triple pole

then $\underset{z=a}{\text{Res}} = \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2} (z-a)^3 z^{n-1} \bar{f}(z)$

Find $z^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right]$ by using Residue theorem

$$z^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right]$$

$$\bar{f}(z) = \frac{2z^2+4z}{(z-2)^3}$$

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$$z^{n-1} \bar{F}(z) = \frac{2z^{n+1} + 4z^{n}}{(z-2)^3}$$

$z = 2$ is a triple pole.

$$\underset{z=2}{\text{Res}} \; z^{n-1} \bar{F}(z) = \frac{1}{2!} \lim_{z \to 2} \frac{d^2}{dz^2} (z-2)^3 \left[\frac{2z^{n+1} + 4z^{n}}{(z-2)^3} \right]$$

$$= \frac{1}{2!} \lim_{z \to 2} \frac{d}{dz} \left[2(n+1) z^{n} + 4n z^{n-1} \right]$$

$$= \frac{1}{2!} \lim_{z \to 2} \left[2(n+1) n z^{n-1} + 4n(n-1) z^{n-2} \right]$$

$$= \frac{1}{2!} \left[2(n+1) n 2^{n-1} + 4n(n-1) 2^{n-2} \right]$$

$$= \frac{1}{2!} \left[(n+1) n 2^{n} + n(n-1) 2^{n} \right]$$

$$= \frac{1}{2!} n 2^{n} \left[(n+\cancel{1}) + (n-\cancel{1}) \right]$$

$$= \frac{1}{\cancel{2}} n 2^{n} \cancel{2} n$$

$$= 2^{n} n^{2}$$

By cauchy's residue theorem

$z^{-1}[\bar{F}(z)]$ = sum of residue of $z^{n-1}[\bar{F}(z)]$

$f(n) = 2^{n} n^{2}$ (Ans)

Find the $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ by using residue.

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

$$F(z) = \frac{z^2}{(z-a)(z-b)}$$

$$z^{n-1}F(z) = \frac{z^{n+1}}{(z-a)(z-b)}$$

$z=a$ is a single pole

$z=b$ is a single pole.

$$\operatorname*{Res}_{z=a} z^{n-1}F(z) = \lim_{z\to a} \cancel{(z-a)} \frac{z^{n+1}}{\cancel{(z-a)}(z-b)}$$

$$= \lim_{z\to a} \frac{z^{n+1}}{z-b}$$

$$= \frac{a^{n+1}}{a-b}$$

$$\operatorname*{Res}_{z=b} z^{n-1}F(z) = \lim_{z\to b} \cancel{(z-b)} \frac{z^{n+1}}{(z-a)\cancel{(z-b)}}$$

$$= \lim_{z\to b} \frac{z^{n+1}}{z-a}.$$

$$= \frac{b^{n+1}}{b-a}$$

By cauchy's residue theorem

$$z^{-1}[\bar{f}(z)] = \text{sum of residue of } z^{n-1}[\bar{f}(z)]$$

$$= \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}$$

$$= \frac{a^{n+1} + b^{n+1}}{a-b}. \quad \text{(Ans)}$$

Applications to find finite difference equation.

Formula:-

1) $z[y_{n+1}] = z\bar{y}(z) - z\,y(0)$

2) $z[y(n+2)] = z^2\bar{y}(z) - z^2 y(0) - z\,y(1)$

3) $z[y_{n+3}] = z^3\bar{y}(z) - z^3 y(0) - z^2 y(1) - z\,y(2)$

4) $z[y_{n-m}] = z^{-m}\,\bar{y}(z)$

1) Solve $y_{n+2} - 4y_{n+1} + 4y_n = 0$ Given that $y(0) = 1$ & $y(1) = 0$

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

$$z[y_{n+2}] - 4z[y_{n+1}] + 4z[y_n] = 0$$

$$z^2 \bar{y}(z) - z^2 y(0) - z\, y(1) - 4[z \bar{y}(z) - z\, y(0)] +$$
$$4 \bar{y}(z) = 0$$

Apply $y(0) = 1$ & $y(1) = 0$

$$z^2 \bar{y}(z) - z^2 - 4z \bar{y}(z) + 4z + 4\bar{y}(z) = 0$$

$$(z^2 - 4z + 4) \bar{y}(z) - z^2 + 4z = 0$$

$$(z-2)^2 \bar{y}(z) = z^2 - 4z$$

$$\bar{y}(z) = \frac{z^2 - 4z}{(z-2)^2}$$

$$\frac{\bar{y}(z)}{z} = \frac{z-4}{(z-2)^2}$$

$$\frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2}$$

$$z - 4 = A(z-2) + B$$

put $z = 2$

$$\boxed{B = -2}$$

Put $z = 0$

$$-4 = -2A + B$$

$$-4 = -2A - 2$$

$$\boxed{A = 1}$$

$$\frac{\bar{y}(z)}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2}$$

$$\bar{y}(z) = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

$$z^{-1}\left[\bar{y}(z)\right] = z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2^n - n2^n$$

$$\boxed{y(n) = 2^n (1-n)}$$

2 Using z- transform solve $y_n - 3y_{n-1} - 4y_{n-2} = 0$

$n \geq 2$ Given that $y(0) = 3$, $y(1) = 2$

$$y_n - 3y_{n-1} - 4y_{n-2} = 0 \qquad \begin{array}{l} y(0) = 3 \\ y(1) = 2 \end{array}$$

Replace $n = n+2$

$$z[y(n+2)] - 3z[y(n+1)] - 4y(n) = 0$$

$$z^2\bar{y}(z) - z^2 y(0) - z\,y(1) - 3[z\bar{y}(z) - z\,y(0)] - 4\bar{y}(z) = 0$$

Apply $y(0) = 3$ & $y(1) = 2$

$$z^2\bar{y}(z) - 3z^2 - 2z - 3z\bar{y}(z) + 9z - 4\bar{y}(z) = 0$$

$$(z^2 - 3z - 4)\bar{y}(z) - 3z^2 + 7z = 0$$

$$(z+1)(z-4)\bar{y}(z) = 3z^2 - 7z$$

$$\bar{y}(z) = \frac{3z^2 - 7z}{(z+1)(z-4)}$$

$$\frac{\bar{y}(z)}{z} = \frac{3z-7}{(z+1)(z-4)}$$

$$= \frac{A}{z+1} + \frac{B}{z-4}$$

$$3z-7 = A(z-4) + B(z+1)$$

put $z=4$

$5 = 5B$

$\boxed{B=1}$

put $z=-1$

$-10 = -5A$

$\boxed{A=2}$

$$\frac{\bar{y}(z)}{z} = \frac{2}{z+1} + \frac{1}{z-4}$$

$$\bar{y}(z) = \frac{2z}{z+1} + \frac{z}{z-4}$$

$$z^{-1}\left[\bar{y}(z)\right] = z^{-1}\left[\frac{2z}{z+1}\right] + z^{-1}\left[\frac{z}{z-4}\right]$$

$$= 2z^{-1}\left[\frac{z}{z+1}\right] + z^{-1}\left[\frac{z}{z-4}\right]$$

$$= 2(-1)^n + (4)^n$$

$$= 4^n + 2(-1)^n \quad \text{(Ans)}$$

16 Solve $x(n+2) - 5x(n+1) + 6x(n) = 5^n$ Given that

$x(0) = x(1) = 0$.

$x(n+2) - 5x(n+1) + 6x(n) = 5^n$

$Z(x(n+2)) - 5Z(x(n+1)) + 6Z\, x(n) = 5^n z$

$z^2 \bar{x}(z) - z^2 x(0) - z\, x(1) - 5[z \bar{x}(z) - z\, x(0)] + 6\bar{x}$

$= \frac{z}{z-5}$

apply $x(0) = x(1) = 0$

$z^2 \bar{x}(z) - 5z\bar{x}(z) + 6\bar{x}(z) = \frac{z}{z-5}$

$(z^2 - 5z + 6)\,\bar{x}(z) = \frac{z}{z-5}$

$(z-3)\,(z-2)\,\bar{x}(z) = \frac{z}{z-5}$

$\bar{x}(z) = \frac{z}{(z-3)(z-2)(z-5)}$

$\therefore$

$z^{n-1}\bar{x}(z) = \frac{z^n}{(z-3)(z-2)(z-5)}$

$z = 3, 2, 5$ is a simple pole.

$\underset{z=2}{Res} = \lim_{z \to 2} \cancel{(z-2)} \frac{z^n}{\cancel{(z-2)}(z-3)(z-5)}$

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$= \frac{2^n}{(-1)(-3)}$

$\underset{z=3}{Res} = \lim_{z \to 3} \cancel{(z-3)} \frac{z^n}{\cancel{(z-3)}(z-2)(z-5)}$

$= \frac{3^n}{(1)(-2)}$

$= -\frac{1}{2} 3^n$

$\underset{z=5}{Res} = \lim_{z \to 5} \cancel{(z-5)} \frac{z^n}{(z-3)(z-2)\cancel{(z-5)}}$

$= \frac{5^n}{(3)(2)}$

$= \frac{1}{6} 5^n$

By cauchy's Residue theorem

$z^{-1} \bar{x}(z) =$ Sum of residue $z^{n-1} \bar{x}(z)$

$x(n) = \frac{2^n}{3} - \frac{3^n}{2} + \frac{5^n}{6}$ (ans).

Formation of Difference Equation

From the difference equation $y_n = a + b3^n$

$y_n = a + b3^n$ —— ①

$y_{n+1} = a + b3^{n+1}$

$y_{n+1} = a + 3b3^n$ —— ②

$y_{n+2} = a + b3^{n+2}$

$y_{n+2} = a + 9b3^{n}$ — (3)

To eliminate a & b

From (1), (2) & (3)

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$6y_n - 8y_{n+1} + 2y_{n+2} = 0$

$3y_n - 4y_{n+1} + y_{n+2} = 0$ (Ans)

From a difference equation by eliminating the arbitary constant a & b from $y_n = a - b3^n$

$y_n = a - b3^n$ — (1)

$y_{n+1} = a - b3^{n+1}$

$y_{n+1} = a - 3b3^{n}$ — (2)

$y_{n+2} = a - b3^{n+2}$

$y_{n+2} = a - 9b3^{n}$ — (3)

To eliminate a & b

From (1), (2) & (3).

$$\begin{vmatrix} y_n & 1 & -1 \\ y_{n+1} & 1 & -3 \\ y_{n+2} & 1 & -9 \end{vmatrix} = 0$$

$$6y_n - 8y_{n+1} + 2y_{n+2} = 0$$

$$3y_n - 4y_{n+1} + y_{n+2} = 0$$

Derive the difference equation from $y_n = (A+Bn)2^n$

$$y_n = (A+Bn)2^n$$

$$y_n = A2^n + Bn2^n \quad \text{——} (1)$$

$$y_{n+1} = A2^{n+1} + B\, 2^{n+1}(n+1)$$

$$y_{n+1} = 2A2^n + 2B(n+1)2^n \rightarrow (2)$$

$$y_{n+2} = A2^{n+2} + B(n+2)2^{n+2}$$

$$y_{n+2} = 4A2^n + 4B(n+2)2^n \rightarrow (3).$$

To eliminate A & B from (1), (2), (3).

$$\begin{vmatrix} y_n & 1 & 1n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$\Rightarrow y_n[8(n+2) - 8(n+1)] - y_{n+1}[4(n+2) - 4n] + y_{n+2}[2(n+1) - 2n] = 0$$

$$8y_n - 8y_{n+1} + 2y_{n+2} = 0$$

$$4y_n - 4y_{n+1} + y_{n+2} = 0$$

Problem Based on Time Sequences

1. Find the Z transform of t

$$Z[t] = Z[nT]$$

$$= T[Z(n)]$$

$$= T\frac{z}{(z-1)^2}$$

$$Z[t] = \frac{Tz}{(z-1)^2} \quad \text{(Ans)}$$

[or]

$$Z[(t)] = \sum_{n=0}^{\infty} nT z^{-n}$$

$$= T/z + 2T/z^2 + 3T/z^3 + \cdots$$

$$= T\left[1/z + 2/z^2 + 3/z^3 + \cdots\right]$$

$$= T/z\left[1 + 1/z + 2/z^2 + \cdots\right]$$

$$= T/z\left[1 - 1/z\right]^{-2}$$

$$= T/z\left[\frac{z-1}{z}\right]^{-2}$$

$$= T/z\left[\frac{z^2}{(z-1)^2}\right]$$

$$Z[t] = \frac{Tz}{(z-1)^2} \quad \text{(Ans)}$$

2. find the z transform of t^2.

$$z[t^2] = z[n^2T^2]$$
$$= T^2 z[n^2]$$
$$= T^2 \frac{z(z+1)}{(z-1)^3} \quad (\text{Ans})$$

3. find the z transform of t^3.

$$z[t^3] = z[n^3T^3] \rightarrow (1)$$

$$z[n^3] = z[n\, n^2]$$
$$= -z\frac{d}{dz} z[n^2]$$
$$= -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right]$$
$$= -z\left[\frac{(z-1)^3(2z+1) - (z^2+z)\,3(z-1)^2}{(z-1)^6}\right]$$
$$= -z(z-1)^2\left[\frac{(z-1)(2z+1) - 3(z^2+z)}{(z-1)^6}\right]$$
$$= -\frac{z}{(z-1)^4}\left[2z^2 - 2z + z - 1 - 3z^2 - 3z\right]$$
$$= -\frac{z}{(z-1)^4}\left[-z^2 - 4z - 1\right]$$

$$z[n^3] = \frac{z(z^2+4z+1)}{(z-1)^4} \rightarrow (2)$$

Sub (2) in (1)

$$z(t^3) = T^3 z[n^3]$$
$$z[n^3] = T^3\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right] \quad (\text{Ans})$$